Universität Duisburg-Essen
Fakultät für Mathematik

The Information Premium on Electricity Markets

A New Spot-Forward Relationship for non-Storable Underlyings

Dissertation

zur Erlangung des Doktorgrades Dr. rer. nat.
der Fakultät für Mathematik
der Universität Duisburg-Essen

vorgelegt von

Richard Biegler-König, M.Sc.

Essen, im Februar 2013

1. Gutachter: Prof. Dr. Rüdiger Kiesel
2. Gutachter: Prof. Dr. Fred Espen Benth

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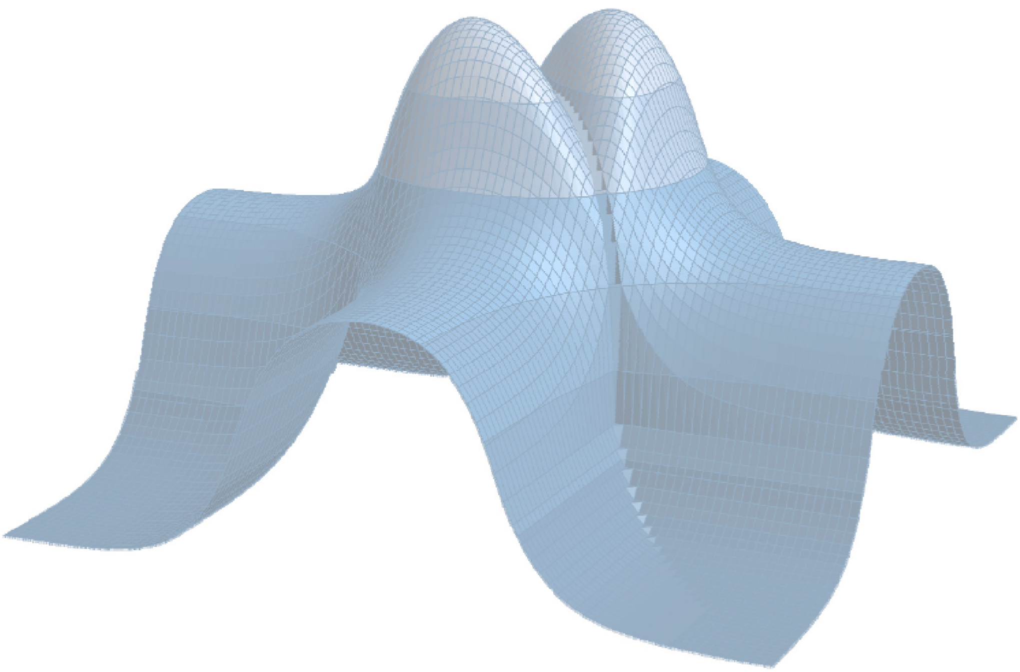

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Amtierender Dekan: Prof. Dr. Ulrich Görtz

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Richard Biegler-König
Essen, April 2013

Abstract

With the deregulation of electricity markets and the establishment of electricity exchanges Quantitative Financial Modelling in this area has become increasingly important. There are two main objects traded on such markets: the spot (usually a day-ahead price) and futures and forwards with various delivery periods. These are settled against the spot. It is thus essential to relate spot and forward prices. In the literature this is usually done by making use of the classical spot-forward relationship which gives the forward price as the risk-neutral expectation of the spot price under the historical filtration. Still, electricity is special in the way that it is non-storable and thus additional information influences future prices while leaving spot prices unaffected. Consequently, the traditional relationship fails theoretically as well as practically. Therefore, this thesis proposes a new extended spot-forward relationship for electricity that does not only take risk-adjustment into consideration but also the so-called market filtration. This is constructed by enlarging the historical filtration with relevant future information. Furthermore, building on the ideas of the reference paper Benth and Meyer-Brandis (2009) we quantify the impact of this approach by means of the information premium, i.e. the difference between the forward price under the market filtration and the traditional forward price.

In the first chapters of this thesis we lay the foundations for the consequent analysis. In the first chapter we motivate the new relationship discussing two EEX market scenarios: the introduction of the second trading phase of the European emission certificates in 2008 and the German Atom Moratorium of 2011. Both exhibit large information premia. The second chapter then discusses the (later to be used) mathematical theory of the enlargement of filtration while the third chapter introduces a popular stochastic spot model. This model will be our work horse for the analytical calculations of this thesis.

The main part of the thesis contributes to the academic literature by analysing various aspects of the impact the new spot-forward relationship has on Financial Modelling. Analytically, we show how to use the mathematical theory to calculate closed-form expressions for the information premium for different types of future information about both the spot and correlated processes. Furthermore, we explore pricing of derivatives on forward contracts in the presence of future information. We also calculate indifference prices and market power of different market traders. Empirically, we provide the first thorough investigation of the interplay of a non-storable commodity and additional future information. We propose a statistical test for the existence of the information premium. We exemplify by analysing the two market situations mentioned above. Our test is based on Hilbert-basis representation and can be more generally applied to assess measurability of two time series.

Summarising, we advocate the necessity and show the feasibility of using a new spot-forward relationship in the context of Quantitative Financial Modelling of electricity markets.

Zusammenfassung

Durch die in den letzten Jahren stattgefundene Deregulierung des Elektrizitätsmarktes und die Gründung von Strombörsen steigt die Relevanz des finanzmathematischen Modellierens in diesem Bereich. Der Handel an solchen Börsen findet zumeist am Spot- und Terminmarkt statt. Während der Spot ein Day-ahead Preis ist, sind die wichtigsten Produkte am Terminmarkt Forwards und Futures mit verschiedenen Lieferperioden. Diese werden gegen den Spotpreis abgerechnet, so dass das Aufstellen eines möglichst präzisen Verhältnisses zwischen beiden Produkten von großer Bedeutung ist. In der akademischen Fachliteratur wurde bislang allerdings ausschliesslich das klassische Spot-Forward Verhältnis übernommen. Dieses gibt den Preis des Forwards als risikoneutralen Erwartungswert bezüglich der natürlichen (historischen) Filtration an. Nun ist jedoch elektrischer Strom insofern ein besonderes Underlying, als dass er nicht speicherbar ist. Zur Verfügung stehende Informationen über die zukünftige Entwicklung werden selbstverständlich zukünftige Preise beeinflussen, nicht aber momentane Preise. Deshalb ist das klassische Spot-Forward Verhältnis weder theoretisch noch praktisch zur Modellierung geeignet.

In dieser Arbeit wird deshalb ein neues, erweitertes Spot-Forward Verhältnis präsentiert, das nicht nur die Risikoanpassung der Marktteilnehmer berücksichtigt, sondern auch die sogenannte Marktfiltration. Diese entsteht durch Vergrößerung der natürlichen Filtration um relevante zusätzliche Zukunftsinformationen. Bezugnehmend auf das Referenzpaper Benth and Meyer-Brandis (2009) werden hier die Auswirkungen dieses neuen Verhältnisses mit Hilfe der Informationsprämie quantifiziert. Diese ist als Differenz zwischen dem Forwardpreis bezüglich der vergrößerten Filtration und demjenigen bezüglich der natürlichen Filtration definiert.

Im ersten Kapitel dieser Arbeit werden zwei EEX Marktszenarien diskutiert, die zur empirischen Motivation für das erweiterte Spot-Forward Verhältnis dienen. Auf der einen Seite ist dies die Einführung der zweiten Phase der europäischen Emissionszertifikate Anfang 2008, auf der anderen Seite das deutsche Atom Moratorium des Jahres 2011. Für beide Szenarien finden wir signifikante Informationsprämien. Das zweite Kapitel beschreibt die später verwendeten mathematischen Grundlagen, insbesondere die Theorie der Filtrationsvergrößerung. Im dritten Kapitel stellen wir ein populäres Spotmodell vor, das im weiteren Verlauf dieser Arbeit Verwendung finden wird.

Im Hauptteil der vorliegenden Arbeit werden die zentralen Resultate vorgestellt. Die Arbeit trägt zur akademischen Literatur durch die Diskussion verschiedener finanzmathematischer Aspekte des erweiterten Spot-Forward Verhältnisses bei. Wir zeigen analytisch, wie die mathematische Theorie angewendet werden kann, um geschlossene Ausdrücke für die Informationsprämie zu finden und zwar sowohl mit Informationen über den zukünftigen Spotpreis, als auch mit Informationen, die korrelierte Prozesse betreffen. Des Weiteren widmen wir uns der Frage, wie zusätzliche Informationen über die zukünftige Entwicklung die finanzmathematische Preisfin-

dung von Derivaten auf Forwards beeinflussen. Außerdem berechnen wir Indifferenzpreise von Forwards für verschiedene Typen von Händlern und ziehen Rückschlüsse auf ihre jeweilige Marktmacht.

Empirisch wird in dieser Arbeit die erste grundlegende Untersuchung zum Zusammenspiel eines nicht speicherbaren Underlyings und dem Markt zur Verfügung stehender Informationen über die zukünftige Entwicklung durchgeführt. Um die Existenz der Informationsprämie zu zeigen, stellen wir einen empirischen Test vor, den wir auf die beiden obigen Marktszenarien anwenden. Dieser Test basiert auf Hilbertraumrepräsentationen und ist generell auch zur Feststellung der Messbarkeit zweier Zeitreihen anwendbar.

Zusammenfassend propagieren wir in dieser Arbeit die Notwendigkeit, im Kontext der finanzmathematischen Modellierung von Elektrizitätsmärkten ein neues, erweitertes Spot-Forward Verhältnis zu verwenden und verdeutlichen die analytischen und empirischen Möglichkeiten, die sich hieraus ergeben.

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Chapter 1.

Introduction and Summary of the Thesis

Although usually characterised as a commodity, electricity is special, in particular when considered as an underlying for financial markets. It possesses a number of properties that influence its use on the one hand and market design on the other hand. The most striking and influential one of these features is its non-storability (at least in relevant quantities). Once electricity has been produced, it needs to be consumed. It is this unique property and its interconnection with the principles of Financial Mathematics that will motivate the theoretical analysis as well as the empirical studies in the dissertation at hand.

This introductory chapter will lay the foundation and provide general information for the dissertation. In Section 1.1 we will briefly introduce electricity markets. This will include an overview of the historical development as well as a description of relevant issues of market design. Furthermore, we will provide a brief summary of the academic literature in the fields of Financial Modelling and Financial Mathematics.

Section 1.2 will then motivate the main subject of this dissertation, the impact of a non-storable underlying when it comes to the relationship between the basic objects on financial markets. This motivation will also be illustrated by making use of two market scenarios that will be discussed in much detail over the course of this dissertation.

In Section 1.3 we will state and formulate the objects of the dissertation and briefly summarise its contribution to the academic literature.

An overview of the structure of this dissertation as well as important information on data sources, computer programming and the publications that form the basis of some chapters and the results therein will finally be brought forward in Section 1.4.

1.1. Electricity Markets

1.1.1. History of Electricity Markets

For more than 100 years electricity has been indispensable for the welfare of modern societies. This is the reason why its generation, distribution and retailing used to be organised by integrated monopolists and in close collaboration with state holdings or local authorities (cf. Ströbele et al. (2010, Section 11.2)). In the 1990s, deregulation laws were passed in a number of developed countries and electricity exchanges were established. This was to allow for more competition and a higher degree of efficiency. In Norway, the *Nord Pool* was founded in 1993 and later joined by Sweden as well as Denmark and Finland (cf. Burger et al. (2007, page 33 ff.)). After a European Union directive (96/92/EC, cf. Ströbele et al. (2010, page 208)) various other European exchanges were started, for example, for Central Europe, the *EEX* in Leipzig in

2002 (by the merger between the *Leipzig Power Exchange* and the *European Energy Exchange*). Other European exchanges include the *APX* (Netherlands) or the *APX UK* (Great Britain). In the USA, the *PJM* (the Pennsylvania-New Jersey-Maryland market) has been influential and trend-setting.

In this thesis the EEX will be our reference market and we will consider EEX time series as well as EEX products. The EEX is the largest power exchange in Europe and together with its subsidiaries (in particular *EPEX* for spot markets) it covers energy and electricity trading in France, Germany, Austria and Switzerland. In 2011 more than 1000 TWh were traded on its derivatives platform and around 300 TWh on the spot market (most of it in Germany, we refer to the annual report EEX (2012b, page 88)). For a comparison, the overall amount of generated electricity in Germany was around 600 TWh (cf. Bundeswirtschaftsministerium (2012, Section Energieträger)).

Amongst other European exchanges, the EEX is also a market place for European emission certificates (EUAs). Following the *Kyoto protocol* of 1997, the *European Union Emissions Trading Scheme* (EUETS) was introduced as a market-based means to reduce the emission of greenhouse gases (see Benth et al. (2008b, page 16)). The first (trial) period of the EUETS started in 2005 and ended in 2007. The second period commenced in 2008 and came to an end in 2012. For these periods, certificates were initially allocated to participants and were valid for one time period. Generally, they allow the holder to emit one tonne of CO_2.

1.1.2. Market Design and Products

In this section we will briefly discuss the objects traded on typical electricity exchanges in general and the EEX in particular. Furthermore, we will only consider those products and those properties that will be relevant over the course of this thesis.

For the EEX area, as mentioned above, spot trading is organised by the EPEX. Although intra-day trading exists, spot prices under consideration here are day-ahead base prices. Prices for individual hours of the following day are calculated by matching bid- and offer-curves (we will comment on this process later). The so-called Phelix day base price is then the simple average of the prices of the 24 hours (i.e. base as well as peak load hours, we refer to booklet EPEX Spot (2012) for details). It is this day-ahead base price that will be called spot price in this thesis. The importance of the spot thus constructed is that it constitutes the reference price for the derivatives market.

The most important products in terms of liquidity and quantity on electricity exchanges are futures contracts. For Germany, these are called Phelix futures on the EEX. These are contracts that are settled either physically or in cash. As electricity is a flow commodity, futures contracts have a delivery period rather than a delivery date. On the EEX delivery periods available are the current and subsequent four weeks, the current and the subsequent nine months as well as various quarters and years (cf. EEX (2012c) for details). Hence, futures contracts on electricity markets have a swap-like structure. Other derivatives traded on the EEX are various forms of options. The underlying of these options mostly are futures contracts as described above, thus they are not written directly on the spot. In the empirical parts of this thesis, we will mainly examine the current and the next six month-futures contracts (these being most liquidly traded).

The number of market players on the EEX is relatively low when compared to traditional markets. The annual report EEX (2012b) gives their number at around 200 participants registered at both the EPEX and the EEX. Clearly, for more specialised products this already hints towards liquidity issues and, in fact, derivatives other than futures are highly illiquid.

The difference between futures and the more basic forward contracts is that the latter is an agreement between two parties whereas the former is organised by an exchange and features a margining process. It is also well-known that one can consider these two related products to be equivalent in case interest rates are deterministic. In this thesis, we will follow the standard approach of the literature and assume this equivalence. This will ease things from a modelling perspective. Furthermore, concerning the data used in the empirical parts of the dissertation, the assumption of deterministic interest rates is not unrealistic, firstly because time horizons under consideration will usually be less than half a year and secondly because interest rates have generally been very low over the last years.

Concerning the matching of demand and capacity on electricity exchanges such as the EEX we still need to mention the *merit order* and we refer to von Roon and Huck (2010) for more details. The merit order is the list of available power plants, sorted in ascending order by their individual marginal costs. The most expensive power plant needed to satisfy demand then sets the price. For the German market this plant is usually either fired with hard coal or gas.

1.1.3. Modelling Electricity Markets - Literature Classification

In this section we will provide a brief overview of the different branches of the scientific literature on modelling electricity markets. We will also classify the approach chosen in this thesis.

The first basic decision that has to be made is whether one models the spot price or the price of forwards. The latter approach is closely connected to the *Heath-Jarrow-Morton* framework known in Financial Mathematics from the field of interest rate modelling. These models have the advantage that one can commence pricing of derivatives directly, rather than having to calculate forward prices first. Important examples of this branch of the literature are Clewlow and Strickland (1999), Börger et al. (2009) or Benth and Koekebakker (2008).

The other possible modelling approach is to construct a model for the spot price and then derive forward prices using some relationship. There are two very different branches of spot models. The first one is that of structural or fundamental models. Here, one takes into consideration driving factors of the electricity spot such as demand, capacity or fuel prices (gas, coal). One then deduces a price for electricity from these factors. Some influential examples of this approach are Aïd et al. (2009), Coulon and Howison (2009) and Burger et al. (2004).

The second approach is that of reduced-form models. Here, one chooses some stochastic model for the spot price directly. This is the typical approach in Financial

Mathematics and well-known from the famous Black-Scholes world. One model of this type will be the workhorse of this thesis and we will provide a detailed literature overview for this class in Chapter 3. The reason why we choose this approach is that in this thesis we propose a certain fundamental relationship between spot and forward prices.

1.2. Spot-Forward Relationships and the Information Approach

After having presented the necessary background information we are now going to motivate the subject of this thesis. We have seen in the previous section that electricity forward contracts are settled against the spot price. Hence, it is natural to relate spot and forward prices. Section 4.2 will feature a detailed discussion about this relationship on electricity markets as well as the corresponding literature. Here, we will discuss in a motivating manner only. We will use the following notation:

Notation 1.2.1. Spot price and forward contract. Let S_t denote the spot price at time t. Furthermore let the forward price at t with maturity at time T be denoted by $F(t,T)$.

In probabilistic terms and by the principle of *risk-neutrality*, the well-known (classical) spot-forward relationship is given by

$$F(t,T) = \mathbb{E}^{\mathbb{Q}}[S_T \mid \mathcal{F}_t] \tag{1.1}$$

Here, $\mathbb{Q}$ is a risk-neutral probability and $\mathcal{F}_t$ is the historical filtration of the spot process (i.e. the filtration generated by past and present values of the spot). As the electricity market is not complete (we will state reasons later), we conclude by the *second fundamental theorem of asset pricing* (cf. Bingham and Kiesel (2004, Theorem 4.3.1) or Shreve (2004, page 232)) that the measure need not be unique. One can choose the specific measure $\mathbb{Q}$ and usually a parametric measure is chosen that minimises the distance between observed forward prices and prices calculated.

Furthermore, the expectation in Equation 1.1 is taken under the historical filtration, i.e. given the information of the spot price evolution thus far. This is usually justified by assuming the validity of the (weak) *efficient market hypothesis* (cf. Hull (2008, page 780)) which says that current (spot) market prices reflect all publicly available information. As stated by Benth and Meyer-Brandis (2009, page 112), the relationship requires the total available information to be equal to the flow of information generated by the spot price process, an assumption acceptable for classical, traded assets.

The relationship is then proved by setting up a replicating portfolio for a short position in a forward. Obviously, this portfolio consists of borrowing money and going long in the underlying. At maturity one can then hand over the underlying and repay the loan. This type of hedging strategy is called *buy-and-hold* strategy.

Equation 1.1 is the relationship used in the quantitative literature on energy markets and not much thought has been invested to whether it constitutes the right approach to relate spot and forward prices of electricity. Even more so, it is used despite a very serious shortcoming: the most important intrinsic (and quite unique) property of electricity as an underlying is that it is not storable (in relevant quantities). Thus, Equation 1.1 fails theoretically as well as practically. Practically, it is not possible to set up the buy-and-hold strategy to hedge an electricity forward. We cannot store spot electricity until maturity. Theoretically, considering the spot market isolated from forward and derivatives markets, the efficient market hypothesis is not valid either. Current prices might not reflect available information about the future. To illustrate this, consider a power generating company announcing the maintenance of one of their power plants in two months' time. If the hypothesis held, then the decrease in production capacity should lead to an increase in the spot price, not only in two months but also immediately. But no arbitrage strategy is available. The spot price will not react to this information. Still, the use of the historical filtration in Equation 1.1 does imply the assumption that current and past spot prices contain all the relevant pricing information. Other stylised examples might include the announcement that a new plant will go online or non-typical and severe weather forecasts or new regulatory legislation. Summarising, the historical filtration is not sufficient when setting up a spot-forward relationship for a non-storable underlying.

Building on Benth and Meyer-Brandis (2009), we will therefore propose a new pricing relationship for electricity forwards:

$$F(t,T) = \mathbb{E}^{\mathbb{Q}}[S_T \mid \mathcal{G}_t] \tag{1.2}$$

where $\mathcal{G}_t$ will be called the *market filtration*. This filtration will contain all the information relevant and publicly available. Thereby, Equation 1.2 will make sure that the forward price will adjust to situations such as described in the stylised example above. Consequently, we will define the *information premium* in Section 4.2 as the difference between the prices as calculated by Equation 1.2 and Equation 1.1. The information premium will be our main object to quantify the influence of the additional information.

Summarising, we postulate that for a valid spot-forward relationship we need to consider not only the risk attitudes of market participants (in terms of a risk-neutral measure) but also the larger information set that is available to them.

In this thesis, and in particular in the empirical chapters (for example Chapter 8), we will justify and illustrate this information-based approach with two market situations from the German EEX market. Both situations feature significant price movements due to forward-looking information. They will be introduced and discussed next.

1.2.1. The Beginning of the Second Phase of the EUETS

After the first, rather non-committal phase of the *European Union Emissions Trading Scheme*, the stricter second phase commenced in January 2008. This phase lasted until 2012. We refer to Section 1.1 for a short market description. In Figure 1.2.1 forward prices observed on 01/10/2006 are illustrated. The delivery period is depicted as the length of the horizontal lines which denote the price. The typical shape of prices in winter is readily observable: lower values for October and April, highest prices in January and February. Slightly lower prices in December are due to clustering of bank holidays.

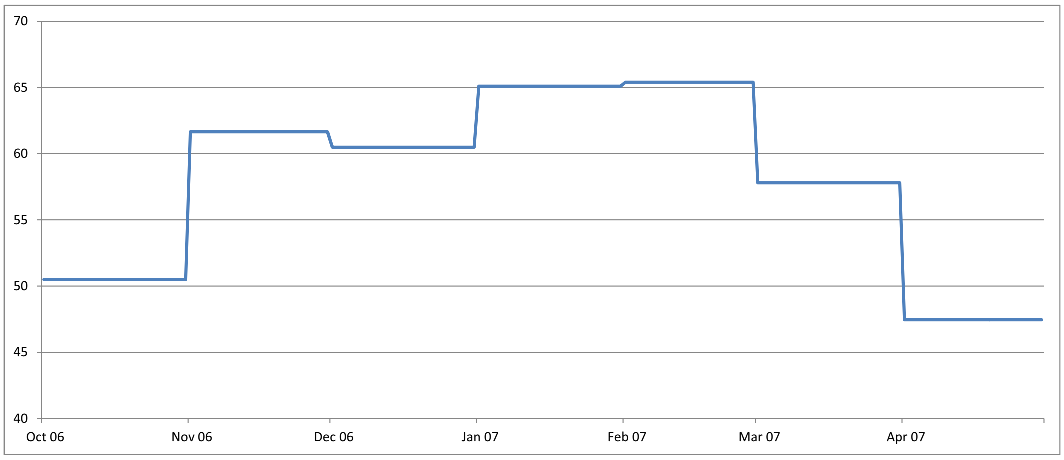


Figure 1.2.1.: EEX forward prices on 01/10/2006. Observed month forward prices in Euro. Length of bars indicates delivery period.

As illustrated in Figure 1.2.2 (which is similar to Figure 1 of Benth and Meyer-Brandis (2009)), we are facing a different situation in the subsequent year. Although all effects seen in Figure 1.2.1 persist, the most striking feature is the huge price increase between the December and the January forwards. The amount of the increase is more than 16 €, corresponding to some 34%, compared to an increase of 4.50 € (around 7.5%) in 2006. Generally, there is a shift upwards in forward prices in 2008. The spot price on 01/10/2007 was around 45 €.

Clearly, the price shift can be explained by the market's anticipation of the effects due to the introduction of the second phase of CO_2 certificates. The costs of these certificates were obviously assumed to cause a major rise in electricity prices. We will provide a quantitative analysis and further qualitative insights in Section 8.4.1.

Summarising, we claim that we find forward prices anticipating publicly available future information but at the same time experience a spot that does not.

1.2.2. The German Atom Moratorium

The Tōhoku earthquake occurred on 11 March 2011. The consequent tsunami severely damaged several nuclear power plants, in particular that in Fukushima. Only three days later, on 14 March 2011, the German government reevaluated its energy policy and issued the so-called *Atom Moratorium* by which the seven oldest plants (eight reactors with a capacity of more than eight GW) were to be shut down immediately (over the course of that week). The measure was to last for a period of three months and was to allow for a new evaluation of the usage of nuclear power in Germany.

Figure 1.2.3 shows the EEX spot price (seven days moving-average for clarity) for the period from November 2010 until mid August 2011. Key dates are highlighted by vertical bars: the earthquake on 11/03/2011, the Moratorium on 14/03/2011, the government's decision to permanently shut down the power plants on 31/05/2011 and the official end of the Moratorium on 15/06/2011. We remark that the vertical

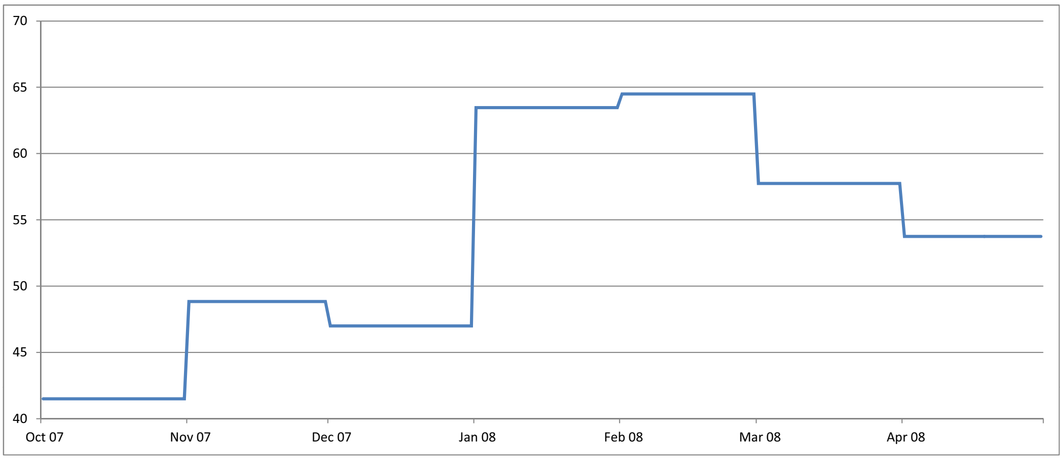


Figure 1.2.2.: EEX forward prices on 01/10/2007. Observed month forward prices in Euro. Length of bars indicates delivery period.

bars will feature throughout the remainder of the thesis.

It is surprising to see that although eight GW of cheap nuclear power were taken from the market (and the merit order), spot prices did not increase or even move significantly. We will find reasons for this and provide a qualitative discussion in Section 8.4.2. We also refer the reader to the official report of the German BNA (Bundesnetzagentur, the federal regulatory office for electricity) to the federal ministry of economics and technology (Bundesnetzagentur (2011)).

Still, the effect of the Moratorium was a sharp increase in forward prices, not only of those whose delivery fell into the three months of the Moratorium but also of those with a later delivery period. As an example, Figure 1.2.4 shows the evolution of the price of the forward with maturity in May 2011. The forward price had a mean value of 46.93 € before the Moratorium and a 57.83 € post-Moratorium mean price. This corresponds to an increase by more than 10 €, i.e. almost 25%. For the second half of this time series we also see that prices remained more or less constant until the last day of the delivery period. Clearly, this implies that by then also the spot had adjusted to the increased price level. Again, we refer to Section 8.4.2 for a discussion and more insights.

Concluding, we find that once more forward prices reacted to some future information which was publicly available but the spot did not (instantly). There is an apparent asymmetry in prices and a large information premium added to forwards by market participants.

1.3. Objectives and Contribution of the Thesis

Electricity markets have only been established relatively recently and are still transforming. Quantitative Financial Modelling of these markets is a challenging task. This is not only due to their incompleteness and illiquidity or due to regulatory issues typical of such markets but also because the underlying electricity is special.

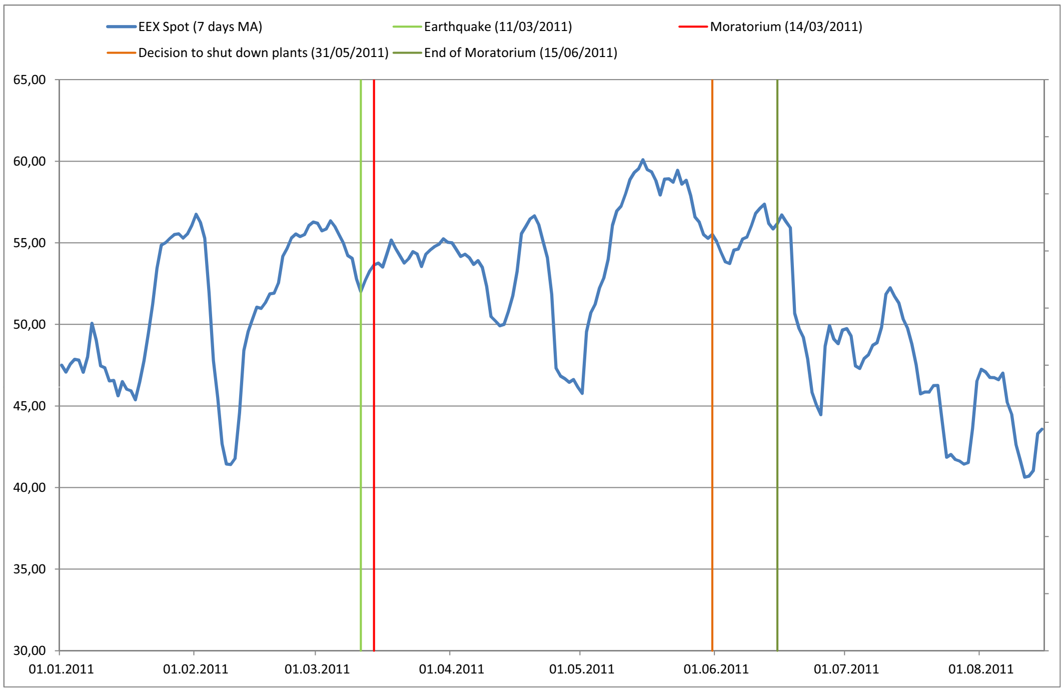


Figure 1.2.3.: EEX spot price 2011/2012. Seven days moving-average from November 2010 until mid of August 2011. In Euro.

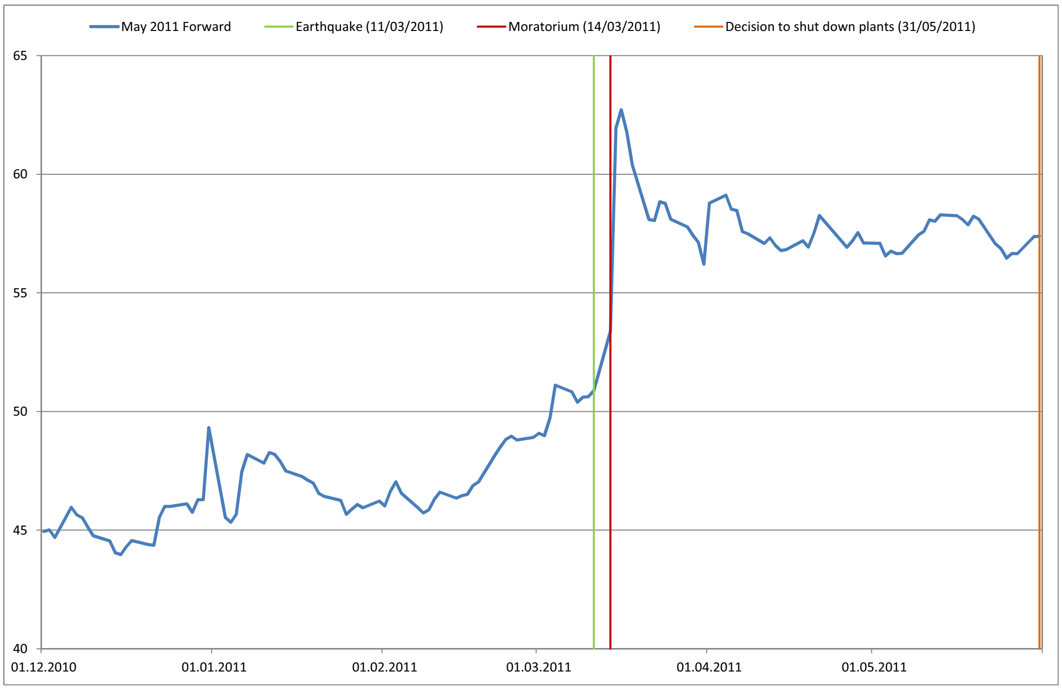


Figure 1.2.4.: EEX May 2011 forward contract. Lifetime from 01/12/2010 until 31/05/2011. In Euro.

It is a flow commodity that cannot be stored in relevant quantities and that needs to be consumed as produced. All types of market players face enormous risks because the market is very volatile and investments needed are usually huge and long-lasting. Forward and futures contracts are the main building blocks of any hedging or trading strategy and their correct valuation as well as their relationships to other basic products is absolutely essential for the industry.

Hence, it is the objective of this thesis to quantitatively investigate the impact of the non-storability of electricity. In particular, we want to explore how the information asymmetry that was motivated in the previous section affects the Financial Mathematics and Financial Modelling of electricity markets. This very important aspect has until now been almost completely neglected in the academic literature. All relevant papers, may they be on pricing, hedging or risk-management in electricity markets, have so far worked with the traditional spot-forward relationship. We thus formulate an extension to the classical relationship of Financial Mathematics and augment the existing literature by discussing various analytical, conceptual as well as empirical consequences that arise from this new basic market structure. We are now going to present the solutions and contributions presented in individual chapters of this thesis one by one.

Building on the reference paper Benth and Meyer-Brandis (2009) and the motivation of Section 1.2 we propose the new spot-forward relationship in Chapter 4. This relationship does not only take a risk-adjustment (via a measure change) into consideration but also accounts for the information asymmetry mentioned before (via an enlargement of filtration).

In Chapter 5 we then extend the analytical results of the reference paper to the more realistic case of forward contracts with a delivery period. In particular, we redefine the object of the information premium for these contracts and calculate it using the mathematical technique of enlargement of filtration. By translating results from the literature on modelling of insider trading we also propose a simpler method to calculate the emerging information drifts making use of Malliavin calculus. Furthermore, we provide formulae for the information premium for various different arrangements of the time axis with one or more pieces of future information and for direct as well as for correlated future information. We also recover the results of Benth and Meyer-Brandis (2009) as limits of our more general formulae.

The issue of pricing options on electricity forwards with delivery period in the presence of an information premium (i.e. under an enlarged filtration) is then discussed in Chapter 6. We identify prices for vanilla options taking the fundamental relationship between the two filtrations at hand into consideration. These prices differ from those calculated in the absence of information asymmetry. Thereby, we show that previous results from the literature do not translate automatically to energy markets. Still, we find that differences in option prices are not due to a modified volatility structure but only due to initial forward prices. Referencing the relevant literature we find that this is a general result that also applies to more complicated pay-off structures.

Motivated by the ideas of the second reference paper Benth, Cartea, and Kiesel (2008a) we identify indifference forward prices of market agents within the above information framework in Chapter 7. This implies calculating the expected utility of the spot under an enlarged filtration. In the existing (and mathematically related) literature on insider trading, authors have thus far always chosen a combination of a log-normal underlying and the logarithmic utility. This leads to a simplifying cancellation and closed-form results. We make use of the distributional properties of the Brownian bridge to obtain results for a more general class of utility functions. As to our best knowledge this is a new approach.

Chapter 8 is the empirical centrepiece of the dissertation at hand. It contributes to the academic literature on electricity markets by providing the first empirical investigation about the information premium and also by showing its existence (and thus the practical relevance of this thesis) for the two scenarios previously discussed in Section 1.2. To this end we propose a statistical method that tests for the properties of the information premium. This turns out to be non-trivial. Without further assumptions on the structure of the enlarged filtration this problem translates to showing that the premium is non-measurable with respect to the historical filtration. In other words, we need to show that the premium is the residual when projecting the forward price under the enlarged filtration onto the space spanned by the historical filtration. Therefore, we propose a method based on Hilbert space representations and basis regressions in order to calculate and analyse expectations under the enlarged filtration. This method can be applied more generally, and to the best of our knowledge no other test for non-measurability has yet been proposed in the literature.

Summarising, we add to the scientific literature by presenting and exploring a new spot-forward relationship for electricity and by providing an overview of how it affects the different aspects and branches of Quantitative Financial Modelling.

1.4. Structure of the Thesis

The structure of this thesis is motivated by the objectives and contributions brought forward in Section 1.3. It is depicted graphically in Figure 1.4.5 which also illustrates the interdependence of the individual chapters.

After the general background information and motivating thoughts of this chapter, Chapter 2 will be devoted to the mathematical theory applied later. In Section 2.2 the theory of enlargement of filtrations will be treated. This will also be the basis for all theoretical chapters of the thesis. Section 2.3 on the other hand will very briefly remember some of the key ideas and results of Hilbert space theory, in particular when applied to spaces of square-integrable stochastic processes.

Then, we will present a certain spot model in Chapter 3. We will calculate forward prices with delivery period for this model in Section 3.4. Furthermore, we will lay the foundations for empirical examinations by describing how to fit the model to observed data (Section 3.5) and how to simulate price paths (Section 3.6). Section 3.7 will provide a first short empirical study.

After discussing spot-forward relationships and the reference paper Benth and Meyer-Brandis (2009) in Chapter 4, we will show how to apply the techniques of the second chapter in order to calculate the information premium for various setups in

Chapter 5. Chapter 6 will discuss the issue of pricing options on futures with additional information. We will extend the ideas of Benth, Cartea, and Kiesel (2008a) and calculate indifference prices and market power in Chapter 7.

Finally, Chapter 8 will feature the thorough empirical study of the information premium, in particular analysing the two market scenarios presented previously. This chapter will bring together the mathematical structures of Chapter 2, the statistical techniques of Chapter 3 and the calculations of Chapter 5.

1.4.1. Data, Programming and Computer Algebra

The price data used in this thesis was obtained from various sources. EPEX Phelix spot and EEX Phelix futures prices were obtained partially from *Bloomberg* and *Thomson Reuters* terminals, partially from the EEX website directly. The EEX gas prices discussed in Section 8.5.1 were taken from a Thomson Reuters terminal and so was the DAX time series of the same section.

Data processing and rearranging was done mainly using *MS Excel* and *MS Visual Basic*. More complicated programmes such as the calibration, simulation and measure change procedures, regressions and statistical tests were written or implemented in *Insightful S-Plus 6.1*. All the graphics in this thesis were prepared using *MS Excel*.

Some complicated analytical expressions were calculated or double-checked making use of two computer algebra systems, namely *Wolfram Mathematica 7* and *WolframAlpha* (online at `http://www.wolframalpha.com`).

1.4.2. Publications

We have highlighted those chapters that are based on publications and working papers in Figure 1.4.5. A version of the discussion on option pricing, which is Chapter 6 in this thesis, is going to be published as part of a Springer proceedings titled *Quantitative Energy Finance*. The title of the chapter will be *Electricity Options and Additional Information* and its reference here is Benth, Biegler-König, and Kiesel (2013b). Parts of the empirical study of Chapter 8 will be published in volume 36 of the journal *Energy Economics* under the title *An Empirical Study of the Information Premium on Electricity Markets*. In this thesis, this paper will be referenced as Benth, Biegler-König, and Kiesel (2013a). Both papers are also available online on SSRN and can be found at `http://papers.ssrn.com/sol3/cf_dev/AbsByAuth.cfm?per_id=1872119`. Last but not least, the material presented in Chapter 7 is the basis of a working paper. Its preliminary title is *Future Information and a Broader Class of Utility Functions* and it will include an additional section about the optimal portfolio of an insider on a stock exchange.

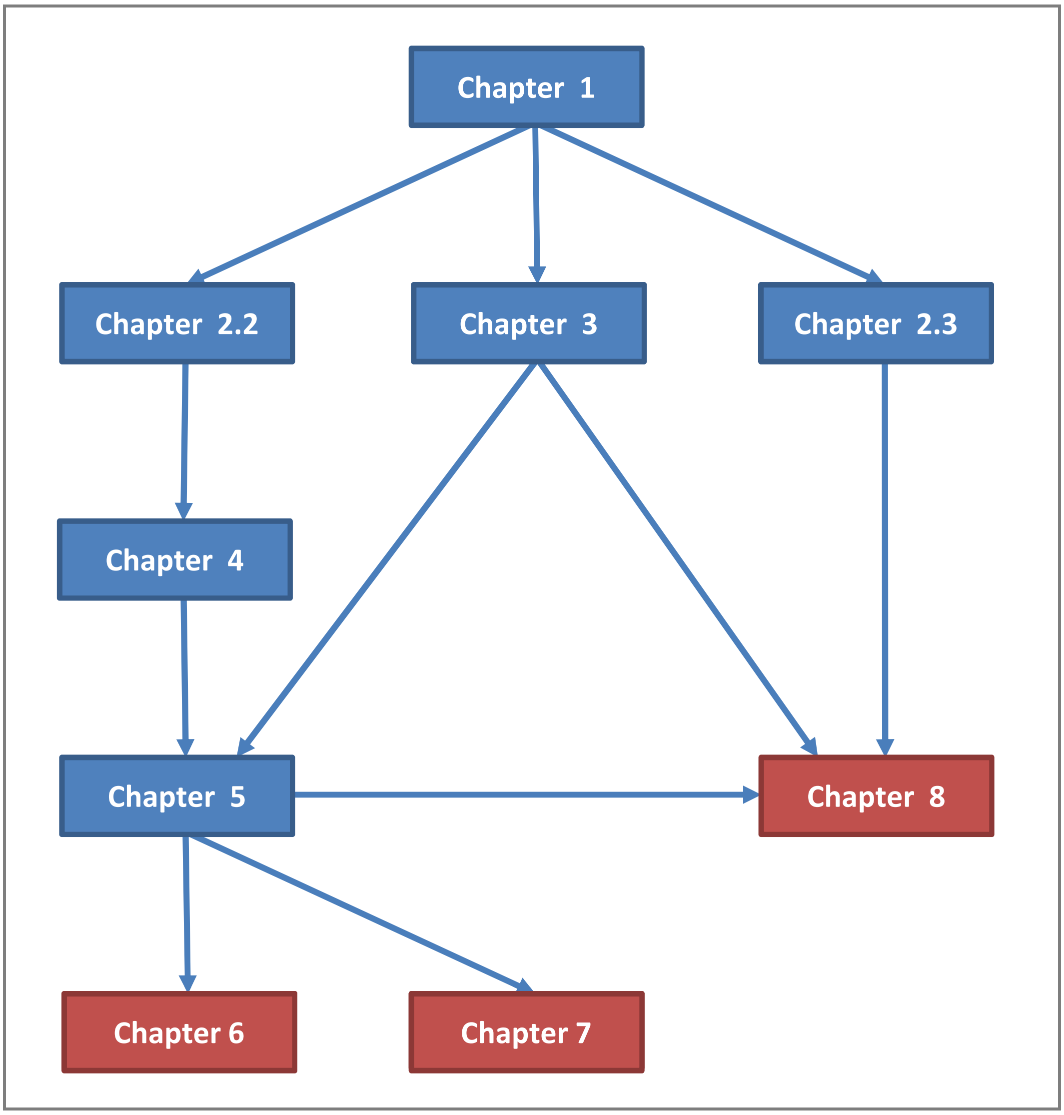


Figure 1.4.5.: Structure of the thesis. Chapters based on publications and working papers highlighted.

Chapter 2.

Mathematical Background

2.1. Literature Overview and Summary

In this chapter we will lay the mathematical foundations for the remainder of the thesis. The first two sections will discuss the *theory of enlargement of filtrations* or French *grossissement de filtrations*.

Historically, research on enlargement of filtrations was initialised by Itō in a contribution to the proceedings Itō (1978). There, he presented the following equation in terms of a Brownian motion W_t, which he said should hold 'from the intuitive meaning of integrals' (Itō (1978, page 95)):

$$\int_s^t W_1 dW_u = W_1(W_t - W_s) \tag{2.1}$$

He realised that the integral on the left side is problematic as W_1 is random for $t < 1$. Thus, in order for the equation to make sense, Itō proposed to consider it under a different filtration (or, as he writes, a different *reference set*), namely the one generated by W_t and W_1, $0 \leq t < 1$. Moreover, he found out that under this filtration the Brownian motion W_t is a semi-martingale, too, and he provided its decomposition. These two results will be crucial in this chapter and we will formulate them as Theorem 2.2.1.

Much scientific progress was then made by French mathematicians in the 1980s. We will talk about a number of relevant papers and proceedings later on, in particular in Section 2.2.1: Chaleyat-Maurel and Jeulin (1985), Jacod (1979), Jacod (1985), Jeulin (1979), Jeulin (1980), Jeulin and Yor (1978), Yor (1978), Yor (1985). Chapter VI of Protter (2005) is a comprehensive reference explaining both branches of the theory: *initial* and *progressive enlargement* (this thesis will only feature the former type). A new motivation to study the enlargement of filtrations was provided by Pikovsky and Karatzas (1996), who applied the theory to model insider trading on financial markets. Since then, a multitude of papers has covered this idea and we will mention the most important ones in the introduction of Chapter 6.

The structure of Section 2.2 will then be as follows: Section 2.2.1 will feature important assumptions and results and we will concentrate on the martingale decomposition of stochastic processes under enlarged filtrations. These decompositions will later be used, for example in Chapter 5.

Then, in Section 2.2.2 we will approach filtration enlargement from a different angle and we will explain how one can consider it as a change of measure. Our main reference for this approach will be the short note by Protter (1989) as well as Föllmer and Imkeller (1993). The doctoral thesis by Ankirchner (2005) also follows the measure change idea. We will utilise the findings of this section in Chapter 6.

Furthermore, we will present Imkeller's method and provide a brief overview of *Malliavin calculus* in Section 2.2.3. This method will greatly simplify our calculations in Chapter 5 although its original motivation was an extension of the French literature. Imkeller established his results in Imkeller (1996, 2003) as well as in Imkeller et al. (2001). The main references on Malliavin calculus we will use are Øksendal (1996) and Nualart (2006).

Yet another aspect of enlargement of filtration will be explored in Section 2.2.4: its connection to linear stochastic differential equations. The well-known *Brownian bridge* will be used to exemplify and to prepare for the more complicated calculations of Chapter 7. The main reference of this section will be Karatzas and Shreve (1991).

Finally, in the last section of this chapter, Section 2.3, we will remind the reader of a number of standard results about Hilbert space theory and its connection to conditional expectations. In particular, we will establish the existence of complete orthonormal systems for the space of square-integrable random variables and cite results that show that conditional expectations on this space possess functional forms. These theoretical results will then be utilised when proposing our empirical test in Chapter 8. Throughout the section we will refer to standard literature such as Royden (1968) for Hilbert space aspects or Klenke (2006) for notions of probability theory.

2.2. Enlargement of Filtration

In Equation 1.2 we proposed a new spot forward relationship in terms of a filtration $\mathcal{G}$ including additional future information. We will now define a general mathematical framework to model this information approach as discussed in Section 1.2. We remark that this is the classical setup as in Jacod (1985, page 15) or Amendinger (1999, page 13).

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space. Throughout this thesis we will always assume that the *usual hypotheses* hold for the spaces we consider and we refer to Protter (2005, Page 3 and Section I.5) for definitions and a discussion. Clearly, $\mathcal{F}_t$ will denote the historical filtration of some underlying stochastic process, say L, i.e. $\mathcal{F}_t = \sigma(L_s : s \leq t)$.

Definition 2.2.1. Enlarged filtration. Let G be an $\mathcal{F}$-measurable random variable with values in a Polish space $(U, \mathcal{U})$. We introduce the (initially) enlarged filtration $\mathcal{H}_t = \mathcal{F}_t \vee \sigma(G)$, $0 \leq t < T_\Upsilon$ where T_Υ denotes the time horizon. Furthermore, we define a filtration $\mathcal{G}_t$ that includes non-precise additional information about G, i.e. $\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{H}_t$.

We remark that we will assume that $\mathcal{H}_{T_\Upsilon} = \mathcal{G}_{T_\Upsilon} = \mathcal{F}_{T_\Upsilon}$, i.e. the additional information expires at time T_Υ. In other words, the additional information concerns the future time point T_Υ. Also, if G was not $\mathcal{F}$-measurable, then the enlarged filtration would be trivial with respect to $\mathcal{F}_t$-measurable processes and the following sections would be obsolete.

2.2.1. Enlargement of Filtration and Martingale Decomposition

In this section we want to answer the question whether or not a semi-martingale under the historical filtration remains a semi-martingale under the enlarged filtration as defined in Definition 2.2.1. If so, we would also like to find its decomposition under the new filtration. Before we answer these questions in general, though, we will go back to Itō's original problem motivated by Equation 2.1. The following famous result provides the answer to both of the above questions for the special case that we enlarge the historical filtration of a process by its future value. We state here a version extended to Lévy processes, the proof will be following the one in Protter (2005, Theorem VI.2.3).

Theorem 2.2.1. Itō's Theorem (extended to Lévy processes). *Let L_t be a Lévy process and $\mathcal{F}_t$ its historical filtration. Let $\mathcal{H}_t = \mathcal{F}_t \vee \sigma(L_{T_\Upsilon})$. Then L is also a semi-martingale with respect to $\mathcal{H}_t$. Also, if $\mathbb{E}[|L_t|] < \infty$, then the process*

$$\xi_t = L_t - \int_0^{t \wedge T_\Upsilon} \frac{L_{T_\Upsilon} - L_s}{T_\Upsilon - s} \, ds$$

is a $\mathcal{H}_t$-martingale on $[0, \infty)$.

Proof. We will prove the statement for $T_\Upsilon = 1$. We can easily generalise due to the scaling properties of Lévy processes. If the statement holds for $t \leq 1$, it holds for $t > 1$ trivially because in this case $\mathcal{F}_t = \mathcal{H}_t$. For simplicity (and without loss of generality) it will be assumed that $\mathbb{E}[L_t] = 0$ and thus L is a $\mathcal{F}_t$-martingale. The proof now consists of three parts:

1. We will temporarily assume that $\mathbb{E}[L_t^2] < \infty$ for all $t > 0$. Now the independent increment property of L is used. One defines auxiliary variables $0 \leq s < t \leq 1$ with $s = \frac{j}{n}$ and $t = \frac{k}{n}$ $(0 \leq j, k \leq n)$ as well as the random variables

$$Y_i = L_{\frac{i+1}{n}} - L_{\frac{i}{n}}$$

yielding the easy identities

$$L_1 - L_s = \sum_{i=j}^{n-1} Y_i \ , \ L_t - L_s = \sum_{i=j}^{k-1} Y_i$$

Also, the length of the interval $1 - s$ in terms of i, j, k, n is $\frac{n}{n} - \frac{j}{n} = \frac{n-j}{n}$, that of $t - s$ is $\frac{k-j}{n}$. Thus, as the Y_i are iid and integrable, we calculate

$$\begin{aligned} \mathbb{E}[L_t - L_s \mid L_1 - L_s] &= \mathbb{E}\left[\sum_{i=j}^{k-1} Y_i \mid \sum_{i=j}^{n-1} Y_i\right] = \frac{n}{n-j} \, \frac{k-j}{n} \, \sum_{i=j}^{n-1} Y_i \\ &= \frac{k-j}{n-j} \, (L_1 - L_s) = \frac{t-s}{1-s} \, (L_1 - L_s) \end{aligned}$$

Clearly, $\mathbb{E}[L_t - L_s \mid \mathcal{H}_s] = \mathbb{E}[L_t - L_s \mid L_1 - L_s]$ for $0 \leq s < t \leq 1$. Hence, verifying the martingale property of the process ξ_t:

$$\begin{aligned}
\mathbb{E}[\xi_t - \xi_s \mid \mathcal{H}_s] &= \mathbb{E}\left[L_t - \int_0^t \frac{L_1 - L_u}{1-u}\, du - L_s + \int_0^s \frac{L_1 - L_u}{1-u}\, du \mid \mathcal{H}_s\right] \\
&= \mathbb{E}\left[L_t - L_s \mid \mathcal{H}_s\right] - \mathbb{E}\left[\int_s^t \frac{L_1 - L_u}{1-u}\, du \mid \mathcal{H}_s\right] \\
&= \frac{t-s}{1-s}\,(L_1 - L_s) - \int_s^t \frac{\mathbb{E}[L_1 - L_u \mid \mathcal{H}_s]}{1-u}\, du \\
&= \frac{t-s}{1-s}\,(L_1 - L_s) - \int_s^t \frac{1}{1-u}\,\frac{1-u}{1-s}(L_1 - L_s)\, du \\
&= 0
\end{aligned}$$

Now, we need the assumption $\mathbb{E}[L_t^2] < \infty$ for the case that $t = 1$ because there is a potential *explosion*, i.e. the process can be unbounded. It needs to be verified that

$$\mathbb{E}\left[\int_0^1 \frac{|L_1 - L_s|}{1-s}\, ds\right] < \infty$$

By using the converse of *Jensen's inequality* and the properties of Lévy processes one gets

$$\mathbb{E}[|L_1 - L_s|] = \mathbb{E}\left[\left((L_1 - L_s)^2\right)^{\frac{1}{2}}\right] \leq \mathbb{E}\left[(L_1 - L_s)^2\right]^{\frac{1}{2}} \leq \kappa(1-s)^{\frac{1}{2}}$$

where κ is some constant. Thus,

$$\mathbb{E}\left[\int_0^1 \frac{\mid L_1 - L_s \mid}{1-s}\, ds\right] \leq \kappa \int_0^1 \frac{(1-s)^{\frac{1}{2}}}{1-s} ds < \infty$$

and the statement is proved.

2. Now, we reduce the assumption $\mathbb{E}[L_t^2] < \infty$ to the weaker statement $\mathbb{E}[|L_t|] < \infty$ for $t > 0$. Now L_t has *càdlàg* paths and thus only a finite number of jumps bigger than 1. Hence, defining

$$\begin{aligned}
J_t^1 &= \sum_{0<s\leq t} \triangle L_s \mathbb{1}_{\{\triangle L_s > 1\}} \\
J_t^2 &= \sum_{0<s\leq t} \triangle L_s \mathbb{1}_{\{\triangle L_s < -1\}}
\end{aligned}$$

we can decompose

$$L_t = Y_t + J_t^1 - J_t^2$$

where Y_t is a L^2-process and independent of J_t^1 and J_t^2. Now, introducing a new filtration $\mathcal{H}'_t = \mathcal{F}_t \vee \sigma(Y_1, J_1^1, J_1^2)$ (which satisfies $\mathcal{H}_t \subset \mathcal{H}'_t$), the process

$$\xi_t = Y_t - \int_0^t \frac{Y_1 - Y_s}{1-s}\, ds$$

is an $\mathcal{H}'_t$ martingale by the first part of this proof. Also, there is no explosion in 1 because neither of the J^i_t jumps at 1 almost surely. For the martingale property of J^i_t we still need to check that

$$\int_0^t \frac{|J^i_1 - J^i_s|}{1-s}\, ds < \infty$$

Both J^i_t are increasing and decreasing, respectively. Also, by stationarity of the increments we can write $\mathbb{E}[J^i_t] = \kappa_i t$. Consequently:

$$\mathbb{E}\left[\int_0^t \frac{|J^i_1 - J^i_s|}{1-s} ds\right] = \left|\int_0^t \frac{\mathbb{E}[J^i_1 - J^i_s]}{1-s} ds\right| = |a_i| \int_0^1 \frac{1-s}{1-s} ds = |a_i| < \infty$$

And Y_t, J^1_t and J^2_t are jointly independent and thus, collecting terms, ξ_t is an $\mathcal{H}'_t$ semi-martingale. *Stricker's theorem* (cf. Theorem A.1) then tells us that it is also an $\mathcal{H}_t$ martingale.

3. Finally, we need to prove that, without any assumption, L is a semi-martingale under the enlarged filtration. We decompose

$$Y_t = L_t - J_t \ , \ J_t = \sum_{0<s\le t} \triangle L_s \mathbb{1}_{\{|\triangle L_s| > 1\}}$$

Now, let $\mathcal{H}^Y_t = \mathcal{F}_t \vee \sigma(Y_t)$, $\mathcal{H}^J_t = \mathcal{F}_t \vee \sigma(J_t)$ and $\mathcal{H}'_t = \mathcal{H}^Y_t \vee \mathcal{H}^J_t$. Then $\mathcal{H}_t \subset \mathcal{H}'_t$. By the first part of this proof, Y_t is an $\mathcal{H}^Y_t$ semi-martingale and it is independent of J_t and thus also an $\mathcal{H}'_t$ semi-martingale and by Stricker's Theorem A.1 an $\mathcal{H}_t$ semi-martingale. J_t on the other hand, is of finite variation and adapted to $\mathcal{H}_t$ and consequently, by Protter (2005, Theorem II.3.7), a semi-martingale. Hence, L_t is an $\mathcal{H}_t$ semi-martingale as well.

This is the end of the proof. □

The following corollary is taken from Nunno et al. (2005, Proposition 5.1) and extends Theorem 2.2.1 to the non-precise filtration $\mathcal{G}_t$.

Corollary 2.2.1. Itō's theorem for non-precise additional information. *The process*

$$\xi_t = L_t - \int_0^t \frac{\mathbb{E}[L_{T_\Upsilon} - L_s \mid \mathcal{G}_s]}{T_\Upsilon - s} ds$$

is a $\mathcal{G}_t$-martingale (for filtration $\mathcal{G}_t$ as defined in Definition 2.2.1).

Proof. For some $u < t$ we have to verify the martingale property:

$$\begin{aligned}
\mathbb{E}[\xi_t \mid \mathcal{G}_u] &= \mathbb{E}\left[L_t - \int_0^t \frac{\mathbb{E}[L_{T_\Upsilon} - L_s \mid \mathcal{G}_s]}{T_\Upsilon - s} ds \mid \mathcal{G}_u\right] \\
&= \mathbb{E}\left[L_t - \int_0^t \frac{L_{T_\Upsilon} - L_s}{T_\Upsilon - s} ds \mid \mathcal{G}_u\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[L_t - \int_0^t \frac{L_{T_\Upsilon} - L_s}{T_\Upsilon - s} ds - L_u + \int_0^u \frac{L_{T_\Upsilon} - L_s}{T_\Upsilon - s} ds \mid \mathcal{H}_u\right] \mid \mathcal{G}_u\right] + \xi_u \\
&= \xi_u
\end{aligned}$$

where in the last step we applied Theorem 2.2.1. □

Theorem 2.2.1 and Corollary 2.2.1 are special cases in which we enlarge by a future value of the process itself and we do not need the distributional properties or more sophisticated machinery for the proof.

Coming back to our (initial) enlargement by some random variable G, we need to answer the question whether a semi-martingale remains a semi-martingale under the enlarged filtration. Jeulin (1980, page 12) formulated the famous hypothesis (H'):

Assumption 2.2.1. Jeulin's Hypothèse (H'). Every $\mathcal{F}$-semi-martingale is a $\mathcal{G}$-semi-martingale.

In Jeulin (1980, Chapitre II) some general results on hypothesis (H') are presented. Furthermore, for specific Brownian frameworks $\mathcal{H}$-decompositions are calculated in Chaleyat-Maurel and Jeulin (1985, Théorème I.1.1) and Jeulin (1980, Chapitre III.3.d). The former decomposition is, in a slightly less general form, used throughout the reference paper Benth and Meyer-Brandis (2009) and we will state it, for completeness, in Theorem A.2.

Here, we will follow the approach of Jacod, presented mostly in Jacod (1985). He builds on the literature mentioned above and formulates the condition (A) (cf. Jacod (1985, page 15)):

Assumption 2.2.2. Jacod's Condition (A). For every t there exists a σ-finite measure η_t on $(U,\mathcal{U})$ such that the regular conditional distribution of random variable G given $\mathcal{F}_t$ is absolutely continuous with respect to η_t for $\mathbb{P}$-almost all $\omega \in \Omega$, i.e. $\mathbb{P}(G \in dl \mid \mathcal{F}_t)(\omega) \ll \eta_t(dl)$.

Jacod (1985, Théorème 1.1) then proves the following theorem. We omit the proof for brevity.

Theorem 2.2.2. Equivalence of conditions (H') and (A). *Assumption 2.2.2 and Assumption 2.2.1 are equivalent.*

Furthermore, he proves that Assumption 2.2.2, in turn, is equivalent to the following much more practical condition (A') (cf. Jacod (1985, Proposition 1.5)) which allows to use the non-conditional distribution of G rather than the process η_t.

Assumption 2.2.3. Jacod's Condition (A'). There exists a σ-finite measure η on $(U,\mathcal{U})$ such that the regular conditional distribution of G given $\mathcal{F}_t$ is absolutely continuous with respect to η for $\mathbb{P}$-almost all $\omega \in \Omega$, i.e. $\mathbb{P}(G \in dl \mid \mathcal{F}_t)(\omega) \ll \eta(dl)$.

The following theorem will provide the equivalence of both conditions and the proof can be found in Jacod (1985, Proposition 1.5).

Theorem 2.2.3. Equivalence of conditions (A) and (A'). *Assumption 2.2.2 holds if and only if Assumption 2.2.3 holds.*

Proof. Clearly, we only need to prove that (A') follows from (A). We let $q_t(\omega)$ be the process that satisfies $\mathbb{P}(G \in dl \mid \mathcal{F}_t)(\omega) = \eta_t(dl)q_t(\omega)$. Then, we define process $p_t(\omega) = \frac{q_t(\omega)}{\mathbb{E}[q_t(\omega)]}$. This is a valid expression as $q_t(\omega) = 0$ a.s. whenever $\mathbb{E}[q_t(\omega)] = 0$. If we now take $\eta(dl) = \mathbb{P}(G \in dl)$ we can calculate for any function $f \in \mathcal{U}$:

$$\int_U f(l)d\mathbb{P}^G = \mathbb{E}[f(G)] = \mathbb{E}\left[\int_U f(l)d\mathbb{P}_t^G\right] = \int_U f(l)\mathbb{E}[q_t]\eta_t(dl)$$

Hence, we have that $\mathbb{P}^G = \mathbb{E}[q_t]\eta_t(l) = \mathbb{E}[q_t]\frac{\mathbb{P}^G_t}{q_t}$ and we can rearrange to get the result that $\mathbb{P}^G_t = p_t\mathbb{P}^G$ which is exactly the statement of condition (A'). □

Two more results describing nice versions of p_t and some regularity conditions are discussed in Jacod (1985, Lemme 1.8, Lemme 1.10). We will not consider them here and proceed to state the main result of this section. This consists of parts (b) and (c) of Théorème 2.5 of Jacod (1985, page 22) and will provide the martingale decomposition of continuous semi-martingales under enlarged filtrations .

Theorem 2.2.4. Jacod's semi-martingale decomposition. *Let M be a continuous $\mathcal{F}_t$-martingale and p_t as defined in the proof of Theorem 2.2.3. Then:*

1. *There exist a previsible process A and a function $(\omega, t, l) \mapsto \kappa^l_t(\omega)$ such that*

$$< p., M. >_t = \int_0^t \kappa^l_s p_{s-} dA_s$$

We can chose $A_t =< M., M. >_t$ if M is also square-integrable.

2. *Furthermore, we have that*

$$\int_0^t |\kappa^L_s| dA_s < \infty \quad a.s. \quad \forall t > 0$$

and the process specified by

$$\xi_t = M_t - \int_0^t \kappa^G_s dA_s = M_t - \int_0^t \frac{d < p., M. >_s}{p_{s-}}$$

is an $\mathcal{H}_t$-semi-martingale.

Proof. We will omit the proof of part (1). For part (2) we will work along the lines of the simplified proof of Jeanblanc (2010, Proposition 2.3.2). Finiteness is trivially true due to the existence of the quadratic variation. To prove the validity of the decomposition we will begin by proving that p_t is an $\mathcal{F}_t$-martingale:

$$\mathbb{E}[p_t \mid \mathcal{F}_s] = \mathbb{E}\left[\frac{\mathbb{P}_t(G \in dl)}{\mathbb{P}(G \in dl)} \mid \mathcal{F}_s\right] = \mathbb{E}\left[\frac{\mathbb{E}[\mathbb{1}_{G\in dl} \mid \mathcal{F}_t]}{\mathbb{E}[\mathbb{1}_{G\in dl}]} \mid \mathcal{F}_s\right] = \frac{\mathbb{E}[\mathbb{1}_{G\in dl} \mid \mathcal{F}_s]}{\mathbb{E}[\mathbb{1}_{G\in dl}]} = p_s$$

Now, let us assume that the $\mathcal{H}$-decomposition of the $\mathcal{F}$-martingale M is given by $M_t = \xi_t + \int_0^t dK_s(G)$. Here, we assume ξ is an $\mathcal{H}$-martingale and thus, by definition of conditional expectation, it should hold that $\mathbb{E}[Xf(G)(\xi_t - \xi_s)] = 0$ for some $\mathcal{F}_s$-measurable variable X and bounded Borel function $f(\cdot)$. Hence, we should have:

$$\begin{aligned}
\mathbb{E}[Xf(G)(M_t - M_s)] &= \mathbb{E}\left[Xf(G)\int_s^t dK_u(G)\right] \\
&= \mathbb{E}\left[X\int_{-\infty}^{\infty} f(l)\int_s^t dK_u(l)p_t(l)\mathbb{P}(G \in dl)\right] \\
&= \int_{-\infty}^{\infty} f(l)\mathbb{E}\left[X\int_s^t \mathbb{E}[p_t(l) \mid \mathcal{F}_u]dK_u(l)\right]\mathbb{P}(G \in dl) \\
&= \int_{-\infty}^{\infty} f(l)\mathbb{E}\left[X\int_s^t p_u(l)dK_u(l)\right]\mathbb{P}(G \in dl)
\end{aligned}$$

where we used stochastic Fubini and in the second but last line the martingale property of p_t. On the other hand, we calculate:

$$\begin{aligned}\mathbb{E}[Xf(G)(M_t - M_s)] &= \mathbb{E}\left[X(M_t - M_s)\mathbb{E}\left[f(G) \mid \mathcal{F}_s\right]\right] \\ &= \mathbb{E}\left[X(M_t - M_s)\int_{-\infty}^{\infty} f(l)p_t(l)\mathbb{P}(G \in dl)\right] \\ &= \int_{-\infty}^{\infty} f(l)\mathbb{E}\left[X(M_t p_t(l) - M_s p_s(l))\right]\mathbb{P}(G \in dl)\end{aligned}$$

Here we used the tower property and that p_t is a martingale. Now, we will examine more closely the inner expectation of that expression. We use the tower property again and condition on $\mathcal{F}_s$ and then apply the *integration-by-parts* formula for semimartingales (cf. Protter (2005, Corollary II.2, page 68)):

$$\begin{aligned}\mathbb{E}\left[X(M_t p_t(l) - M_s p_s(l))\right] &= \mathbb{E}\left[X\mathbb{E}\left[\int_s^t d(M_u p_u(l)) \mid \mathcal{F}_s\right]\right] \\ &= \mathbb{E}\left[X\mathbb{E}\left[\int_s^t d < M_\cdot, p_\cdot(l) >_u + \int_s^t (M_u dp_u(l) + p_u(l)dM_u) \mid \mathcal{F}_s\right]\right] \\ &= \mathbb{E}\left[X\int_s^t d < M_\cdot, p_\cdot(l) >_u\right]\end{aligned}$$

where only the quadratic covariation is left because both expectations of integrals with respect to martingales M_t and $p_t(l)$ are zero. We can substitute back and collect terms:

$$\mathbb{E}[Xf(G)(M_t - M_s)] = \int_{-\infty}^{\infty} f(l)\mathbb{E}\left[X\int_s^t d < M_\cdot, p_\cdot(l) >_u\right]\mathbb{P}(G \in dl)$$

Comparing the last lines of both calculations then yields the equation

$$\int_{-\infty}^{\infty} h(l)\mathbb{E}\left[X\int_s^t (d < M_\cdot, p_\cdot(l) >_u - p_u(l)dK_u(l))\right]\mathbb{P}(G \in dl) = 0$$

and thus

$$dK_u(l) = \frac{d < M_\cdot, p_\cdot(l) >_u}{p_u(l)}$$

which is exactly the decomposition proposed by the statement of the theorem. □

Clearly, Theorem 2.2.4 provides an applicable method to identify the decomposition of martingales under an enlarged filtration. In order to calculate the quadratic variation of part (2) of the theorem all we need to find is the canonical decomposition of the process p_t. This can be done by applying Itō's theorem. For example, if we consider the Brownian motion W_t it turns out that

$$p_t(\cdot, l) = p_0(\cdot, l) + \int_0^t \kappa_s^l p_s(\cdot, l)dW_s \tag{2.2}$$

with the notation of Theorem 2.2.4.

There are two important aspects of Theorem 2.2.4 that still need to be mentioned: Firstly, an extension to non-continuous enlargements is provided in Ankirchner (2008, Theorem 2.3) and would theoretically allow more realistic applications to Lévy processes later. Still, most jump-models (including the one introduced in Chapter 3) do not possess closed-form densities and, thus, this extension does not yield closed-form decompositions. Secondly, we will introduce an object called the *information yield* later (cf. Definition 4.3.1) which will be defined exactly in the same way as the auxiliary process $K_t(l)$ from Theorem 2.2.4.

Now, as for Itō's theorem (i.e. Theorem 2.2.1), we find a corollary for the case that additional information is not precise, meaning for filtration $\mathcal{G}_t$ rather than $\mathcal{H}_t$.

Corollary 2.2.2. Jacod's decomposition for non-precise additional information. *Let M_t be a continuous $\mathcal{F}_t$-martingale and let $\mathcal{G}_t \subseteq \mathcal{F}_t \vee \sigma(G)$. Then process*

$$\xi_t = M_t - \int_0^t \mathbb{E}\left[\frac{d < p., M. >_s}{p_{s-}} \mid \mathcal{G}_t\right]$$

is a $\mathcal{G}_t$-semi-martingale.

Proof. Exactly as in Corollary 2.2.1. □

We need one more rather technical result that will later enable us to change the boundaries of integrals in most situations. This is taken from Benth and Meyer-Brandis (2009, Proposition A.3).

Theorem 2.2.5. Integral boundaries for certain decompositions. *As before, let L_t be a Lévy process, $\mathcal{G}_t \subseteq \mathcal{H}_t = \mathcal{F}_t \vee \sigma(G)$ be the enlarged filtration and the time horizon as before T_Υ. Further, we assume that the decomposition of L_t under $\mathcal{G}_t$ takes the form*

$$L_t = \xi_t + \int_0^t g(s)\mathbb{E}\left[\int_s^{T_\Upsilon} f(u)dL_u \mid \mathcal{G}_s\right] ds$$

where ξ_t is a $\mathcal{G}_t$-martingale and where $g(t)$ and $f(t)$ are continuous functions on $[0, T_\Upsilon]$. Then:

$$\mathbb{E}\left[\int_s^{T_\Upsilon} f(u)dL_u \mid \mathcal{G}_t\right] = \mathbb{E}\left[\int_t^{T_\Upsilon} f(u)dL_u \mid \mathcal{G}_t\right] \exp\left(-\int_t^s f(u)g(u)du\right)$$

for $t \leq s \leq T_\Upsilon$.

Proof. We begin the proof by introducing an auxiliary process Y_s defined by

$$Y_s = \mathbb{E}\left[\int_s^{T_\Upsilon} f(u)dL_u \mid \mathcal{G}_t\right]$$

By the tower property this gives rise to

$$\begin{aligned}
Y_s &= Y_t - \mathbb{E}\left[\int_t^s f(u)dL_u \mid \mathcal{G}_t\right] \\
&= Y_t - \mathbb{E}\left[\int_t^s f(u)\left(g(u)\mathbb{E}\left[\int_u^{T_\Upsilon} f(v)dL_v \mid \mathcal{G}_u\right]\right)du \mid \mathcal{G}_t\right] \\
&= Y_t - \int_t^s f(u)g(u)\mathbb{E}\left[\int_u^{T_\Upsilon} f(v)dL_v \mid \mathcal{G}_t\right]du \\
&= Y_t - \int_t^s f(u)g(u)Y_u du
\end{aligned}$$

and the solution to this integral equation is easily identified:

$$Y_s = Y_t \exp\left(-\int_t^s f(u)g(u)du\right)$$

and this is the required result. □

2.2.2. Enlargement of Filtration and Change of Measure

A different approach to the one presented in the last section is to consider enlargement of filtration as a special type of changing measure. The connection between those two concepts was first discovered in Protter (1989) and indeed we will rely on this short paper in Chapter 6. Here, though, we will briefly introduce the idea using Amendinger (1999) as our main reference (which in turn is based on Föllmer and Imkeller (1993)).

Let a new measure $\mathbb{Q}_{\mathcal{F}}$ be specified by its density process Z_t. This will be the martingale measure under the historical filtration. The following lemma is a version of Proposition 1.6 of Amendinger (1999).

Lemma 2.2.1. Construction of the decoupling measure. *Here, we assume a slightly stricter version of Assumption 2.2.3 due to Föllmer and Imkeller (1993) which says that the regular conditional distribution given $\mathcal{F}_t$ is not only absolutely continuous but also equivalent to the law of G. Using the notation of Section 2.2.1 the following statements hold:*

1. *The process $\frac{Z_t}{p_t^G}$ is a $(\mathbb{P}, \mathcal{G})$-martingale.*

2. *The measure $\mathbb{Q}_{\mathcal{G}}$ defined by $\frac{d\mathbb{Q}_{\mathcal{G}}}{d\mathbb{P}}\mid_t = \frac{Z_t}{p_t^G}$ decouples $\mathcal{F}_t$ and $\sigma(G)$, i.e. they are independent under $\mathbb{Q}_{\mathcal{G}}$. In particular, this means that for $A_t \in \mathcal{F}_t$ and $B \in \mathcal{U}$ the following holds:*

$$\mathbb{Q}_{\mathcal{G}}(A_t \cap \{G \in B\}) = \mathbb{Q}_{\mathcal{F}}(A_t)\mathbb{P}(G \in B) = \mathbb{Q}_{\mathcal{G}}(A_t)\mathbb{Q}_{\mathcal{G}}(G \in B)$$

Proof. We start by proving the first equation of (2), calculating

$$\begin{aligned}
\mathbb{Q}_{\mathcal{G}}(A_t \cap \{G \in B\}) &= \mathbb{E}\left[\frac{Z_t}{p_t^G}\mathbb{1}_{A_t\cap\{G\in B\}}\right] \\
&= \mathbb{E}\left[Z_t \mathbb{1}_{A_t}\mathbb{E}\left[\mathbb{1}_{\{G\in B\}}\frac{1}{p_t^G} \mid \mathcal{F}_t\right]\right] \\
&= \int_{A_t} Z_t(\omega)\mathbb{E}\left[\mathbb{1}_{\{G\in B\}}\frac{1}{p_t^G} \mid \mathcal{F}_t\right](\omega)\mathbb{P}(d\omega) \\
&= \int_{A_t} Z_t(\omega)\int_B \frac{1}{p_t^l}p_t^l \mathbb{P}(G\in dl)\mathbb{P}(d\omega) \\
&= \int_{A_t} Z_t(\omega)\mathbb{P}(d\omega)\mathbb{P}(G\in B) \\
&= \mathbb{Q}_{\mathcal{F}}(A_t)\mathbb{P}(G\in B)
\end{aligned}$$

The second equation can be derived when setting in turn $A_t = \Omega$ or $B = \mathcal{U}$. For (1) we check the martingale property by using the same ideas as above as well as the fact that Z is a Radon-Nikodym derivative and thus a martingale. Let $s < t$.

$$\begin{aligned}
\mathbb{E}\left[\frac{Z_t}{p_t^G}\mathbb{1}_{A_s\cap\{G\in B\}}\right] &= \mathbb{E}\left[\mathbb{E}\left[Z_t \mid \mathcal{F}_s\right]\mathbb{1}_{A_s}\mathbb{E}\left[\frac{1}{p_t^G}\mathbb{1}_{\{G\in B\}} \mid \mathcal{F}_t\right]\right] \\
&= \mathbb{E}\left[Z_s \mathbb{1}_{A_s}\int_B \frac{1}{p_t^G}p_t^G d\mathbb{P}\right] \\
&= \mathbb{E}\left[Z_s \mathbb{1}_{A_s}\int_B \frac{1}{p_s^G}p_s^G d\mathbb{P}\right] \\
&= \mathbb{E}\left[\frac{Z_s}{p_s^G}\mathbb{1}_{A_s\cap\{G\in B\}}\right]
\end{aligned}$$

Also, it is obvious that $p_0^G = 1$ as well as that $Z_0 = 1$ which proves that $\mathbb{Q}_{\mathcal{G}}$ is a valid measure. □

Now, using Lemma 2.2.1, we can show the following theorem, stating that the martingale property is preserved under enlargement of filtration when also changing the measure (this is Theorem 1.7 of Amendinger (1999)). Basically, this is the equivalent of Theorem 2.2.3 when approaching enlargement of filtration from a change of measure perspective.

Theorem 2.2.6. Preservation of the martingale property. *Any* $(\mathbb{Q}_{\mathcal{F}}, \mathcal{F})$*-martingale is a* $(\mathbb{Q}_{\mathcal{G}}, \mathcal{G})$*-martingale.*

Proof. Let L be a $(\mathbb{Q}_{\mathcal{F}}, \mathcal{F})$-martingale, let $0 \leq s \leq t$ as well as $A_s \in \mathcal{F}_s$, $B \in \mathcal{U}$. Similar to the proof of Lemma 2.2.1 we calculate:

$$\mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}\left[\mathbb{1}_{A_s\cap\{G\in B\}}L_t\right] = \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}\left[\mathbb{1}_{A_s}L_t\right]\mathbb{Q}_{\mathcal{G}}(G\in B)$$

where we have used the decoupling property of $\mathbb{Q}_{\mathcal{G}}$. Now for the first part

$$\mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}\left[\mathbb{1}_{A_s}L_t\right] = \mathbb{E}^{\mathbb{Q}_{\mathcal{F}}}\left[\mathbb{1}_{A_s}L_t\right] = \mathbb{E}^{\mathbb{Q}_{\mathcal{F}}}\left[\mathbb{1}_{A_s}L_s\right] = \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}\left[\mathbb{1}_{A_s}L_s\right]$$

because $\mathbb{Q}_{\mathcal{G}} = \mathbb{Q}_{\mathcal{F}}$ on $(\Omega, \mathcal{F})$. □

Thus, we have presented enlargement of filtration from a different point of view. Still, it is important to realise that we can go from $\mathcal{G}$ to $\mathcal{F}$ by changing measure but the reverse is not true. Without knowledge of $\mathcal{G}$ the extra information it contains is not attainable from an $\mathcal{F}$-point of view. This will be crucial later on and, in particular, in Lemma 4.2.2.

2.2.3. Malliavin Calculus and Imkeller's Method

Having discussed another approach to the enlargement of filtration we will now return to an extension of the decomposition results as presented in Section 2.2.1. Theorem 2.2.4 postulated that the drift of a continuous martingale under an enlarged filtration is given by

$$dK_t(l) = \frac{d < p.(l), M. >_t}{p_{t-}(l)} \tag{2.3}$$

In practice, in order to calculate this expression we will need to find the dynamics of $p_t(\cdot)$, which can be very tedious as we will show in Section 5.2. Furthermore, the crucial ingredient of Equation 2.3 is the quadratic covariation and this might already hint towards the direction of the *stochastic calculus of variations* or *Malliavin calculus*. Indeed, in a number of papers Peter Imkeller discovered and explored this connection (these are Imkeller (1996, 2003), Imkeller et al. (2001)).

As we will make good use of Imkeller's results in Chapter 5, we will now very briefly remind the reader of those concepts of Malliavin's calculus that we will apply. After that, Theorem 2.2.8 will provide the decomposition of martingales under an enlarged filtration in terms of the *Malliavin derivative*. We have already mentioned the standard literature on Malliavin calculus in Section 2.1 and we will use both Øksendal (1996) and Nualart (2006).

Generally, in Malliavin calculus, one is interested in differentiating a square-integrable random variable $F : \Omega \mapsto \mathbb{R}$ with respect to the chance parameter $\omega \in \Omega$ (cf. Nualart (2006, page 24)). In the following, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{F}_t$ the historical filtration generated by a Brownian motion W_t.

The starting point for almost every discussion about this field of stochastic calculus is the Wiener-Itō chaos expansion. Fix a time horizon, $T > 0$. Then the chaos expansion postulates that every $\mathcal{F}_T$-measurable random variable can be expressed as some sum of iterated Itō integrals. The so-called Malliavin derivative is then defined very naturally for this representation of a random variable (by removing one of the integrals, cf. Nualart (2006, Proposition 1.2.1)).

Still, it turns out that for this thesis it is sufficient to consider a less complicated class of random variables, the *Wiener polynomials*.

Definition 2.2.2. Wiener polynomials. Define random variables $\Theta_i(t) = \int_0^t f_i(s)dW_s$ for functions $f_i(s) \in L^2([0,T])$ and $1 \leq i \leq n$. Furthermore, let $a_i \in \mathbb{R}$ for all $1 \leq i \leq n$. Then the function $\theta(\cdot)$ defined by

$$\theta(\Theta_1(t), \ldots, \Theta_n(t)) = \sum_{i=1}^{n} a_i \Theta_i(t)^i$$

is called a *Wiener polynomial*. Furthermore, we let the space of Wiener polynomials be denoted by $\mathcal{W}$.

Øksendal (1996, page 4.6) stresses that the space $\mathcal{W}$ is dense in $L^2(\Omega)$ and thus it is also sufficient from a theoretical point of view.

The following definition collects and summarises briefly the main objects of Malliavin calculus (cf. Øksendal (1996, Definition 4.6., Definition 4.7., Definition 4.10., Theorem 4.11., Definition 4.13.)):

Definition 2.2.3. Malliavin derivative and Cameron-Martin space.

1. The *Cameron-Martin space* consists of the set of so-called *Cameron-Martin directions* $\gamma(t) \in \Omega$, which can be written as

$$\gamma(t) = \int_0^t g(s)ds$$

for a function $g(t) \in L^2([0,T])$.

2. The *directional derivative* of a random variable $F : \Omega \mapsto \mathbb{R}$ in the direction of γ (if it exists) is defined as

$$D_\gamma F(\omega) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}(F(\omega + \epsilon\gamma) - F(\omega))$$

3. If the directional derivative exists for all γ of the above form (i.e. they are in $L^2(\Omega)$) and if also a process $\psi(t,\omega) \in L^2([0,T] \times \omega)$ exists such that

$$D_\gamma F(\omega) = \int_0^T \psi(t,\omega)g(s)ds$$

exists, then we say that the random variable F is *differentiable* and

$$D_t F(\omega) = \psi(t,\omega) \in L^2([0,T],\Omega)$$

is called the *derivative* of F.

4. We will now denote by $\mathbb{W}$ the closure of the space $\mathcal{W}$ with respect to the norm

$$\| F \|_{1,2} = \| F \|_{L^2(\Omega)} + \| D_t F \|_{L^2([0,T],\Omega)}$$

Leaving out some of the technicalities for the sake of brevity (we refer the reader to Øksendal (1996, Theorem 4.11. and Lemma 4.14.)) we let a series $\{F_n\} \subset \mathbb{W}$ be defined such that $F_n \to F$ in $L^2(\Omega)$. Furthermore, we assume that $\{D_t F_n\}_{n=1}^\infty$ is convergent in $L^2([0,T],\omega)$. Then, we call

$$\mathcal{D}_t F = \lim_{n \to \infty} D_t F_n$$

the *Malliavin derivative* of F. This is uniquely defined and coincides with the derivative as defined in part (3) for every $F \in \mathcal{W}$.

Before we continue, we will illustrate the objects of Definition 2.2.3 by calculating the Malliavin derivative of the simplest Wiener polynomial (cf. Øksendal (1996, Example 4.8.)):

Example 2.2.1. Malliavin derivative of an Itō integral. Let random variable F be given by

$$F = \int_0^T f(s) dW_s$$

where function $f \in L^2([0,T])$. Clearly, we have that $F \in \mathcal{W}$. With the notation of Definition 2.2.3 let us now calculate the directional derivative of F:

$$\begin{aligned} D_\gamma F(\omega) &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(F(\omega + \epsilon\gamma) - F(\omega) \right) \\ &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\epsilon \int_0^T f(s) g(s) ds \right) \\ &= \int_0^T f(s) g(s) ds \end{aligned}$$

Hence, the derivative and the Malliavin derivative are given by

$$D_t F = \mathcal{D}_t F = f(t)$$

As mentioned before, taking the Malliavin derivative removes, so to speak, one stochastic integral.

Having illustrated, defined and ensured existence of the derivative we will now state a result that will be very useful for calculations later. This is the chain rule and provided in Lemma 4.9. of Øksendal (1996):

Theorem 2.2.7. The chain rule of Malliavin calculus. *Let $\theta(\cdot)$ be a Wiener polynomial. Then, for $s < t$ its Malliavin derivative takes the form*

$$\mathcal{D}_s \theta(\Theta_1(t), \dots, \Theta_n(t)) = \sum_{i=1}^{n} \frac{\partial \theta}{\partial \Theta_i(t)} (\Theta_1(t), \dots, \Theta_n(t)) f_i(s)$$

as expected intuitively.

We remark that this result is generalised to continuously differentiable functions with bounded partial derivatives in Nualart (2006, Proposition 1.2.2, Proposition 1.2.3.).

Now that we have introduced the toolbox of Malliavin calculus we will proceed by connecting it to the theory of enlargement of filtrations. In Imkeller et al. (2001) the authors find that Jacod's condition (A) (c.f. Assumption 2.2.2) is too restrictive to deal with enlargement by the largest value of the process under consideration, i.e.

$$\mathcal{H}_t = \mathcal{F}_t \vee \sigma \left(\sup_{s \in [0, T_\Upsilon]} W_s \right) \tag{2.4}$$

The reason for this is that the conditional distribution of the supremum is expressed in terms of a Dirac measure whereas the the unconditional distribution is absolutely continuous with respect to the Lebesgue measure. For a more detailed discussion we refer to Imkeller (2003, page 160). Imkeller and his co-authors then try to circumvent this restriction by applying methods from Malliavin calculus. We will summarise briefly along the lines of page 162 of Imkeller (2003) referring to the sources quoted therein for technical details.

Their starting point is the classical *Clarc-Ocone formula* (c.f. Nualart (2006, Proposition 1.3.5)) which states that suitable random variables F (i.e. those living in the Banach space induced by completion with respect to the norm described in part (4) of Definition 2.2.3) possesses the following decomposition:

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}\left[\mathcal{D}_t F \mid \mathcal{F}_t\right] dW_t \tag{2.5}$$

where $T > 0$ is a finite time horizon and as usual $\mathcal{F}_t$ is the filtration generated by the Brownian motion W_t. Imkeller continues by arguing that the objects under consideration in the classical enlargement of filtration framework are conditional densities. These are martingales and it is thus possible to interchange the expectation and the Malliavin derivative operators. This leads to the following measure-valued version of Equation 2.5:

$$p_t(\cdot, l) = p_0(\cdot, l) + \int_0^t \mathcal{D}_s p_s(\cdot, l) dW_s \tag{2.6}$$

where p_t is defined as in Section 2.2.1, i.e. $\mathbb{P}_t^G = p_t \mathbb{P}^G$. We remark that this is Equation 4.3 in Imkeller (1996) and the derivation is given in detail in Imkeller et al. (2001, page 9 et seq.). But now we can remember the last expression of decomposition Equation 2.2 and equate it with the corresponding part from Equation 2.6:

$$\kappa_t^l p_t(\cdot) = \mathcal{D}_t p_t(\cdot, l) \tag{2.7}$$

Hence, we get another decomposition of a $\mathcal{G}_t$-Brownian motion ξ_t:

$$\xi_t = W_t - \int_0^t \kappa_s^l ds = W_t - \int_0^t \frac{\mathcal{D}_s p_s(\cdot, l)}{p_s(\cdot, l)} ds \tag{2.8}$$

Furthermore, referring to Equation 4.5 of Imkeller (2003), we can interchange the Malliavin operator and the Radon-Nikodým derivative in the definition of p_t yielding the drift

$$\frac{\mathcal{D}_t p_t(\cdot, l)}{p_t(\cdot, l)} = \frac{\mathcal{D}_t \frac{d\mathbb{P}_t^G}{d\mathbb{P}^G}(\cdot, l)}{\frac{d\mathbb{P}_t^G}{d\mathbb{P}^G}(\cdot, l)} = \frac{\mathcal{D}_t \mathbb{P}_t^G(\cdot, dl)}{\mathbb{P}_t^G(\cdot, dl)} \tag{2.9}$$

This basically means that we can now replace the condition that $\mathbb{P}_t^G(\omega, dl)$ has to be absolutely continuous with respect to $\mathbb{P}^G(dl)$ with the following less restrictive one (cf. Imkeller (2003, Equation 4.14)):

Assumption 2.2.4. Imkeller's condition. The Malliavin derivative $\mathcal{D}_t\mathbb{P}_t^G(\cdot, dl)$ is absolutely continuous with respect to $\mathbb{P}_t^G(\cdot, dl)$ $\mathbb{P}$-a.s and $\forall t \in [0, T_\Upsilon]$.

This allows us to finally state the following theorem (without proof, cf. Theorem 4.3 of Imkeller (2003)):

Theorem 2.2.8. Imkeller's method. *Assume Assumption 2.2.4 is satisfied and that*

$$\int_0^t \mid \frac{\mathcal{D}_s\mathbb{P}_s^G(\cdot, dl)}{\mathbb{P}_s^G(\cdot, dl)} \mid ds < \infty$$

Then, the process

$$\xi_t = W_t - \int_0^t \frac{\mathcal{D}_s\mathbb{P}_s^G(\cdot, dl)}{\mathbb{P}_s^G(\cdot, dl)} ds$$

is a $\mathcal{G}_t$-martingale for $l = G$.

We remark that under Assumption 2.2.4 it is possible to consider semi-martingales under the filtration described by Equation 2.4. No relation between conditional and unconditional laws needs to be satisfied. In this thesis, though, we will apply Theorem 2.2.8 not to make specific problems tractable but to avoid very tedious calculations. Comparing Section 5.2.2.1 and Section 5.2.2.2 later on will illustrate the degree of simplification.

2.2.4. Enlargement of Filtration and Linear Differential Equations

In this section we will briefly introduce yet another approach to enlargement of filtration while at the same time preparing for the calculations of Chapter 7. We have seen thus far various ways to find the decomposition of a $\mathcal{G}$-martingale in terms of an $\mathcal{F}$-martingale (we remember Theorem 2.2.8. Corollary 2.2.2 or Theorem 2.2.1). For the remainder of this section and in view of the later application we will from now on only consider a Brownian framework.

Written in a general way and with the usual notation the decompositions of this chapter were all of the following form:

$$d\xi_t = dW_t - f(W_t)dt$$

for some function $f(\cdot)$ to be called the information drift later. Rearranging terms gives

$$dW_t = f(W_t)dt + d\xi_t$$

and we can really also view this as a linear stochastic differential equation in terms of the stochastic process ξ_t. Karatzas and Shreve (1991, Section 5.6) describe generally how to solve such an equation and we will briefly summarise. Partially using their notation the general linear stochastic differential equation is given as

$$dX_t = (A(t)X_t + a(t))\,dt + \sigma(t)d\xi_t\ ,\ X_0 = x \tag{2.10}$$

for a Brownian motion ξ_t and non-random suitable functions $a(t), A(t), \sigma(t)$. Solving this is done by first considering the deterministic differential equation

$$\frac{d\bar{X}_t}{dt} = A(t)\bar{X}_t + a(t) \ , \ \bar{X}_0 = x \tag{2.11}$$

Now, denoting by $\Phi(t)$ the solution to the homogeneous equation $d\Phi(t) = A(t)\Phi(t)dt$ one can identify the solution of Equation 2.11 as

$$\bar{X}_t = \Phi(t)\left(\bar{X}_0 + \int_0^t \Phi^{-1}(s)a(s)ds\right) \tag{2.12}$$

Using Itō's lemma the solution of Equation 2.10 can then be found to be

$$X_t = \Phi(t)\left(X_0 + \int_0^t \Phi^{-1}(s)a(s)ds + \int_0^t \Phi^{-1}(s)\sigma(s)d\xi_s\right) \tag{2.13}$$

and we will discuss a detailed example of this procedure in Section 7.3.1.

Now, what is the advantage of this new interpretation of the decomposition of the Brownian motion under an enlarged filtration? This interpretation allows to find a representation of an $\mathcal{F}$-process X_t only in terms of the $\mathcal{G}$-Brownian motion ξ_t and without dependence on its own value X_s for $0 \leq s \leq t$. And this in turn allows to deduce the distribution of X_t.

As an example, let us look at the well-known *Brownian bridge* (i.e. a Brownian motion on an interval with known endpoint, we refer to Karatzas and Shreve (1991, page 358)):

$$dX_t = \frac{X_{T_\Upsilon} - X_t}{T_\Upsilon - t}dt + dW_t$$

Clearly, this is just Theorem 2.2.1 in disguise. Also, applying the solution method from above, we can postulate the following:

Lemma 2.2.2. Distribution of the Brownian bridge. *Under its own filtration the Brownian bridge is conditionally normally distributed.*

Proof. We refer the reader to Karatzas and Shreve (1991, Lemma 6.9). A more advanced case is presented in Section 7.3.1. □

Having realised the connection between enlargement of filtration on the one hand and stochastic linear differential equation on the other hand will allow some innovative calculations in Chapter 7.

2.3. Conditional Expectation and Hilbert Space Representations

In Chapter 8 we will propose a method to empirically show the existence of the information premium (to be properly defined in Definition 4.2.3 and Definition 5.3.1) which will utilise Hilbert space theory. For this reason we will briefly list some fundamental results in this section and refer the reader to the appropriate scientific literature. For a comprehensive discussion and a definition of Hilbert spaces we refer the reader to Jantscher (1977) (in particular Definition 15.I and 16.I).

The next theorem is taken from Royden (1968, Proposition 10.27, page 212) and gives the most important property of Hilbert spaces, namely that they allow for a representation in terms of an orthonormal system of their elements.

Theorem 2.3.1. Existence of a complete orthonormal system. *In a separable Hilbert space every orthonormal system is countable and there is a complete orthonormal system. Every element of the Hilbert space can be expressed in terms of a complete orthonormal system.*

Over the course of this thesis we will assume that all objects under consideration live in the space of square-integrable (finite-variance) random variables, i.e. they are in $L^2(\Omega, \mathcal{G}, \mathbb{Q})$. Jantscher (1977, page 64) shows that this space is separable. Furthermore, it is also a Hilbert space, see for example Klenke (2006, Korollar 7.22), and complete orthonormal systems exist by Theorem 2.3.1. In Chapter 8 we will try to find the representation of the functional form of a conditional expectation in terms of a basis. Hence, we need the following (quite intuitive) results:

Theorem 2.3.2. Functional form of conditional expectation. *Let $\mathcal{F} \subseteq \mathcal{G}$ be two filtrations and let $X \in L^2(\Omega, \mathcal{G})$.*

1. *Conditional expectation is a contraction:*

$$X \in L^2(\Omega, \mathcal{G}) \Rightarrow \mathbb{E}[X|\mathcal{F}] \in L^2(\Omega, \mathcal{F})$$

   The resulting random variable once again lives in a Hilbert space.

2. *A functional form exists for the conditional expectation.*

Proof. Part (2) stems from the measurability properties of conditional expectation and the factorisation lemma, we refer to Klenke (2006, Korollar 1.97). For part (1) we refer to Malliavin (1995, part (1) of Theorem 2.1.3). □

Chapter 3.

The Arithmetic Spot Price Model

3.1. Literature Overview and Summary

In this chapter, we will present a certain general stochastic model for the spot price of electricity. We will then introduce a more specific version of that model which will be our workhorse for the remainder of this thesis. We will discuss how to calculate forward prices with delivery period under that model as well as how to calibrate it to observed market data. We will provide a description of how to simulate price paths for the model. The chapter will conclude with a demonstration of the model and the techniques presented using observed price data from the German EEX market.

In reduced-form models (cf. Section 1.1.3) commodity spot prices are traditionally modelled using some kind of mean reversion process, the reason being that one assumes prices to fluctuate around their marginal production costs. The most famous and one of the easiest models is the one presented in Schwartz (1997). Here, the log-price of the commodity spot is modelled using a Gaussian Ornstein-Uhlenbeck process. The geometric setup of this model has the obvious advantage of not permitting negative prices. In Schwartz and Smith (2000) this model is then extended to include not only a (short-term) mean reversion factor but also features a long-term equilibrium level modelled using a Brownian motion with drift. This model can be shown to be equivalent to the one proposed in Gibson and Schwartz (1990). Electricity spot prices not only exhibit seasonalities and mean reversion; due to non-storability as well as inflexible capacity and demand they feature jumps and spikes as well (cf. Section 1.1). Hence, the use of Lévy processes has long been advocated and both jump-diffusion as well as pure-jump models have been presented in the literature (examples of the former are Cartea and Figueroa (2005), Eydeland and Wolyniec (2003), Pilipovic (2007), Clewlow and Strickland (2000), Benth et al. (2003), Benth et al. (2010), Eberlein and Stahl (2003), whereas Weron (2006), Benth et al. (2007a), Benth and Ŝaltyte-Benth (2004) feature the latter type of models). Furthermore, negative prices have been observed on electricty markets. These are caused by physical contraints (such as ramp-up and ramp-down times) as well as the political and regulatory framework (for example by the forced infeed of renewables, cf. Section 1.1). This, in fact, makes geometric models and their use (partially) obsolete. Lucia and Schwartz (2002) thus propose the use of arithmetic models instead, mainly because they are usually more tractable. In this chapter, we will work empirically with and discuss analytically a version of the arithmetic model introduced in Benth et al. (2007a). This is further described in Benth et al. (2008b, Section 3.2.2). This model features a deterministic function that is to capture all seasonal effects as well as a sum of Gaussian and non-Gaussian Ornstein-Uhlenbeck processes. Depending on their mean reversion parameters these can be interpreted as short, medium or long-term components. Different versions of this arithmetic spot model are widely used and, for example, form the basis of papers such as Benth et al. (2008a) (cf. Chapter 7), Benth et al. (2009), Benth et al. (2013a) (cf. Chapter 8) or Benth et al. (2013b) (cf. Chapter 6).

In Section 3.4 of this chapter we will calculate forward prices for this model and the main references will be the two papers Benth et al. (2007a) and Benth et al. (2008a). Here, we will consider a special two-factor setup in which one factor (which will be called the base component) will be driven by a Brownian motion whereas the other component (the jump component) will feature the specific Lévy process introduced by Kou (2002). Kou's goal was to model stock prices more realistically than with a *Geometric Brownian motion* (as in the Black-Scholes case) while at the same time maintaining some tractability. Section 3.5 will address the problem of how to estimate the parameters of the model and how to fit it to observed data. The methods then described and applied later in Section 3.7 have partially been taken from different papers such as Cartea and Figueroa (2005), Meyer-Brandis and Tankov (2008) or the two papers Borovkova and Permana (2006) and Borovkova et al. (2009).

We further remark that the model described in this chapter can be used for other underlyings as well. Benth et al. (2008b, Chapter 5) use it to model the logarithm of the gas spot market whereas it is (in a simple form) used in the context of temperature modelling in Benth et al. (2007b) (cf. also Section 5.5).

3.2. Description of the Model

As mentioned above, the spot model used in this thesis will be a reduced-form multi-factor arithmetic model.

We will now formally introduce the model:

Definition 3.2.1. General spot model. Let S_t denote the spot price process at time t. We define

$$S_t = \Lambda_t + \sum_{i=1}^{n} X_t^i$$

Here, Λ_t is the deterministic function capturing predictable seasonal effects. The X_t^i are mean reversion level zero Lévy Ornstein-Uhlenbeck processes solving the stochastic differential equations:

$$dX_t^i = -\alpha_i X_t^i dt + dL_t^i$$

where $\alpha_i \in \mathbb{R}^+$ is the (constant) mean reversion parameter and L_t^i are Lévy processes. We further assume that all L_t^i satisfy the *exponential moment condition*, i.e. that for all i there exist positive constants h_1^i, h_2^i and $h^i \in (-h_1^i, h_2^i)$ such that $\mathbb{E}[e^{h^i L_1^i}] < \infty$ (we refer to Riesner (2003) or standard books on Lévy processes such as Applebaum (2004) for more details).

We remark that it is possible to consider mean reversion parameters depending on time, i.e. $\alpha_i \equiv \alpha_i(t)$. Still, in this thesis, we will only consider constant parameters. A standard application of Itō's lemma (with function $f(x,t) = e^{\alpha_i t}x$) yields, for $T > t$, the solution of the Ornstein-Uhlenbeck processes:

$$X_T^i = e^{-\alpha_i(T-t)} X_t^i + \int_t^T e^{-\alpha_i(T-s)} dL_s^i \tag{3.1}$$

According to the classical spot-forward relationship (see Equation 1.1) we can calculate the forward price as the conditional expectation of the spot process. Knowing for example the moment-generating function of the processes L_t^i it is easy to see that the arithmetic structure of the model will allow for closed-form solutions. In particular, this is true if we consider forwards with a delivery period and thus the expectation of the integral over the spot. This will be further discussed in Section 3.4.1. Still, we will also require the simpler case of a forward with a delivery date for the discussions in Chapter 4. Note that for the moment we choose the real-world measure $\mathbb{P}$ for the valuation. We refer to the discussion in Section 1.2 for more details on the pricing measure.

Proposition 3.2.1. Forward price. *The price of a forward on the spot in t with delivery at T, denoted as $F(t,T)$, is given by:*

$$F(t,T) = \Lambda_T + \sum_{i=1}^{n} e^{-\alpha_i(T-t)} X_t^i - \sum_{i=1}^{n} \frac{1}{\alpha_i} \psi'_{L_1^i}(0) \left(1 - e^{-\alpha_i(T-t)}\right)$$

where $\psi'_{L_t^i}(0)$ is the first cumulant of L_1^i.

Proof. The origin of the first two terms is readily seen when considering Equation 3.1 and the fact that $F(t,T) = \mathbb{E}[S_T|\mathcal{F}_t]$. For the third term, the exponential moment condition guarantees the existence of the decomposition $L_t^i = c_i B_t^i + M_t^i + \mathbb{E}[L_1^i]t$. Here, $c_i \in \mathbb{R}$, B_t^i are Brownian motions and M_t^i are compensated jump components with $\mathbb{E}[M_t^i] = 0 \ \forall t$. Thus:

$$\begin{aligned}
\mathbb{E}\left[\int_t^T e^{\alpha_i(T-s)} dL_s^i \mid \mathcal{F}_t\right] &= \mathbb{E}\left[\int_t^T e^{\alpha_i(T-s)} \mathbb{E}[L_1^i] ds \mid \mathcal{F}_t\right] \\
&= \mathbb{E}[L_1^i] \int_t^T e^{\alpha_i(T-s)} ds \\
&= \tfrac{1}{\beta}\ \mathbb{E}[L_1^i]\ \left(1 - e^{-\beta(T-t)}\right)
\end{aligned}$$

where the expectations of martingales are zero. As Lévy processes are normally specified in terms of the characteristic function $\varphi_{L_1^i}(u) = \mathbb{E}[e^{\imath u L_1^i}]$, we make use of cumulant generating function

$$\psi^k_{L_1^i}(0) = \frac{1}{\imath^k} \frac{\partial^k}{\partial u^k} \ln \varphi_{L_1^i}(u) \mid_{u=0}$$

and the identity $\mathbb{E}[L_1^i] = \psi'_{L_1^i}(0)$. The result follows. □

The intuition behind this setup is that there exist different sources of randomness which have different time horizons. Some events might influence the price of electricity over a longer period (temperature, power plant maintenance) whereas others are rather short-term (very high wind infeed, outage of a plant). Still, it is not sufficient to add more factors but all factors also require their own mean reversion speed. Thinking about price spikes this becomes obvious. This property is satisfied by the model.

All predictable movements in the price will be captured by the seasonality function Λ_t. Normally, one assumes that there exists a slow linear trend in prices, some kind of annual pattern and an adjustment to the weekly patterns. In this thesis we only consider daily data so that Λ_t need not include intra-day patterns. Generally, the more complicated the seasonality function, the greater the danger of overfitting.

The shape of the seasonality function is fairly standard in the literature. In what is to follow we will utilise:

Definition 3.2.2. Seasonality function. The function Λ_t will be given by

$$\begin{aligned}\Lambda_t = b_0 + b_1 t + b_2 \cos(\tfrac{2\pi}{365}t) + b_3 \sin(\tfrac{2\pi}{365}t) + b_4 \cos(\tfrac{2\pi}{182}t) + b_5 \sin(\tfrac{2\pi}{182}t) \\ + b_6 \mathbb{1}_{\{t\ mod\ 7=0\}}(t) + b_7 \mathbb{1}_{\{t\ mod\ 7=1\}}(t) + \ldots + b_{12} \mathbb{1}_{\{t\ mod\ 7=6\}}(t)\end{aligned}$$

where b_0, b_1 are trend parameters, $b_3, \ldots, b_6$ are yearly and half-yearly parameters and $b_7, \ldots, b_{12}$ are dummy parameters that cover the weekdays.

We are now going to introduce a specific version of the arithmetic model that will be our workhorse for the remainder of this thesis.

3.3. Two-factor Model with Double-exponentially Distributed Jumps

We will use the following two-factor and constant-parameter version of the spot model:

Definition 3.3.1. Two-factor model. We consider

$$S_t = \Lambda_t + X_t + Y_t \tag{3.2}$$

where Λ_t is again the deterministic seasonality function and X_t is a Gaussian Ornstein-Uhlenbeck solving

$$dX_t = -\alpha X_t dt + \sigma dW_t$$

with $\alpha, \sigma \in \mathbb{R}^+$ and W_t a standard Brownian motion. Also, Y_t is a Lévy Ornstein-Uhlenbeck process

$$dY_t = -\beta Y_t dt + dL_t$$

where again $\beta \in \mathbb{R}^+$ and L_t is a Lévy process satisfying the exponential moment condition.

In this framework X_t is supposed to model the mid-term behaviour of the spot and we will call it the *base component*. Accordingly, Y_t models short-term spikes. Clearly, we will expect α to be smaller than β. It turns out that two factors are enough to capture most of the properties of the electricity spot price. We remark that with a slightly different setup two factors were also found to be sufficient in Benth et al. (2012).

Furthermore, as mentioned in Section 3.1, we choose a specific process for L_t proposed originally by Kou (2002).

Definition 3.3.2. Jump component. Define the compound Poisson process

$$L_t = \sum_{i=1}^{N_t} D_i$$

where N_t is a Poisson process with intensity λ and the D_i are the jump sizes and *iid*. Furthermore, let the D_i be double-exponentially distributed, i.e. with exponentially distributed positive and negative jumps. The density of the jump sizes is then

$$f_D(x) = p\eta_1 e^{-\eta_1 x}\mathbb{1}_{x\geq 0} + q\eta_2 e^{\eta_2 x}\mathbb{1}_{x\leq 0} \tag{3.3}$$

where p and q are the probabilities of positive and negative jumps and satisfy $p+q = 1$. Parameters $\eta_1, \eta_2 > 0$ are the corresponding intensities.

Equation 3.3 is a valid distribution as can easily be shown by a simple calculation:

$$\int_{-\infty}^{\infty} f_D(x)dx = \int_{-\infty}^{0} q\eta_2 e^{\eta_2 x}dx + \int_{0}^{\infty} p\eta_1 e^{-\eta_1 x}dx = q + p = 1$$

For explicit calculations involving the expectation of the jump component (such as in Proposition 3.2.1) we need the following lemma (see also Papapantoleon (2008, page 34)):

Lemma 3.3.1. Log-moment generating function.

1. *The log-moment generating function of* L_1 *is given by*

$$\psi_{L_1}(u) = \log \mathbb{E}[e^{uL_1}] = \frac{-\lambda p u}{u-\eta_1} + \frac{\lambda q u}{u-\eta_2}$$

2. *Its first and second derivatives are:*

$$\begin{aligned} \psi'_{L_1}(u) &= \frac{-\lambda p}{u-\eta_1} + \frac{\lambda p u}{(u-\eta_1)^2} + \frac{\lambda q}{u-\eta_2} + \frac{-\lambda q u}{(u-\eta_2)^2} \\ \psi''_{L_1}(u) &= \frac{2\lambda p}{(u-\eta_1)^2} - \frac{2\lambda p u}{(u-\eta_1)^3} - \frac{2\lambda q}{(u-\eta_2)^2} + \frac{2\lambda q u}{(u-\eta_2)^3} \end{aligned}$$

3. *And the values at* $u = 0$ *are*

$$\psi'_{L_1}(0) = \frac{\lambda p}{\eta_1} - \frac{\lambda q}{\eta_2} \quad , \quad \psi''_{L_1}(0) = \frac{2\lambda p}{\eta_1^2} - \frac{2\lambda q}{\eta_2^2}$$

Proof. 1. The moment generating function of compound Poisson processes is given by (cf. Applebaum (2004, Section 1.3) or Cont and Tankov (2004, page 74))

$$\mathbb{E}[e^{uL_t}] = \exp\left(t\lambda \int_{\mathbb{R}} (e^{uy} - 1)\, F(dy)\right)$$

where $F(\cdot)$ is the distribution of the jump heights. This allows to calculate the log-moment generating function at $t = 1$ making use of Equation 3.3

$$\begin{aligned}
\psi_{L_1}(u) &= \log\left(\exp\left(\lambda \int_{\mathbb{R}} (e^{uy} - 1)\, F(dy)\right)\right) \\
&= \lambda \int_{\mathbb{R}} (e^{uy} - 1) \left(p\ \eta_1\ e^{-\eta_1 y} \mathbb{1}_{\{y \geq 0\}} + q\ \eta_2\ e^{\eta_2 y} \mathbb{1}_{\{y<0\}}\right) dy \\
&= \lambda \eta_1 p \int_0^{\infty} (e^{uy} - 1)\, e^{-\eta_1 y} dy + \lambda \eta_2 q \int_{-\infty}^{0} (e^{uy} - 1)\, e^{\eta_2 y} dy \\
&= \lambda \eta_1 p \int_0^{\infty} \left(e^{(u-\eta_1)y} - e^{-\eta_1 y}\right) dy + \lambda \eta_2 q \int_{-\infty}^{0} \left(e^{(u+\eta_2)y} - e^{\eta_2 y}\right) dy \\
&= \frac{-\lambda p u}{u - \eta_1} + \frac{\lambda q u}{u + \eta_2}
\end{aligned}$$

The second and third part of the lemma follow immediately. □

We will need one more result in Section 3.4.2 that will enable us to change the measure.

Lemma 3.3.2. Integrability condition. *The compound Poisson process with double-exponentially distributed jump heights as defined in Definition 3.3.2 satisfies the integrability condition*

$$\int_{|x| \geq 1} e^{\theta x} \nu(dx) < \infty$$

for $\theta_L < \eta_1$ *as well as* $\theta_L > -\eta_2$. *Here,* $\nu(dx)$ *is the Lévy measure of* L_t.

Proof. With the density of the jump heights as well as the moment-generating function of L_t given we can calculate straightforwardly:

$$\begin{aligned}
\int_{|x| \geq 1} e^{\theta x} \nu(dx) &= \int_{-\infty}^{-1} e^{\theta x} \lambda q \eta_2 e^{-\eta_2 |x|} dx + \int_1^{\infty} e^{\theta x} \lambda p \eta_1 e^{-\eta_1 x} dx \\
&= \lambda q \eta_2 \int_{-\infty}^{-1} e^{(\theta + \eta_2)x} dx + \lambda p \eta_1 \int_1^{\infty} e^{(\theta - \eta_1)x} dx \\
&= \frac{\lambda q \eta_2}{\theta + \eta_2}\ e^{-\eta_2 - \theta} - \frac{\lambda p \eta_1}{\theta - \eta_1}\ e^{\theta - \eta_1} \\
&< \infty
\end{aligned}$$

because the Lévy measure of a compound Poisson process is given by $\nu(dx) = \lambda f_D(x)$. □

We can illustrate Proposition 3.2.1:

Example 3.3.1. Specific forward price. With the spot model as specified by Definition 3.3.1 and Definition 3.3.2 the forward price $F(t,T)$ is given by

$$F(t,T) = \Lambda_T + e^{-\alpha(T-t)}X_t + e^{-\beta(T-t)}Y_t + \frac{1}{\beta}\left(\frac{\lambda p}{\eta_1} - \frac{\lambda q}{\eta_2}\right)\left(1 - e^{-\beta(T-t)}\right)$$

Here, we have used Lemma 3.3.1 and the fact that the expectation of an Itō integral is zero.

3.4. Forward Pricing with Delivery Period

Electricity forwards/futures do usually not have a single maturity but feature a so-called delivery period. This means that each day in that period a fixed amount of electricity is exchanged at a fixed price. Thus, as mentioned in Section 1.1.2, their payoff structures are really those of swaps and hence the requirement for a spot model to allow closed-form forward prices is quite strict. Furthermore, contracts can be, and are settled financially against the spot price.

In order to find an expression for such a forward/swap we first let T_1 and T_2 denote the start and final date of the delivery period. Following loosely Benth et al. (2008b, Section 1.5.2) and the reasoning leading to Equation 1.1 we can write the payoff of the forward

$$\sum_{t_i=T_1}^{T_2} S_{t_i} - (T_2 - T_1)F(t, T_1, T_2)$$

where the t_i are the hours between T_1 and T_2 (because the spot constitutes, as mentioned in Section 1.1, an hourly time series). From now on, we will use integrals rather than sums, the reason being mathematical convenience. With an interest rate r_t and a risk-neutral measure $\mathbb{Q}$ (again, we refer to Section 1.2 for a discussion on pricing measures on electricity markets) the risk-neutral valuation formula yields:

$$0 = \mathbb{E}^{\mathbb{Q}}\left[\int_{T_1}^{T_2} e^{-r(u-t)}(S_u - F(t,T_1,T_2))du|\mathcal{G}_t\right]$$

where $\mathcal{G}_t$ is the filtration containing the information used to calculate the price. We will always assume that settlement of forwards will take place at T_2, i.e. at the end of the delivery period. Hence, the discounting factor cancels:

$$0 = \mathbb{E}^{\mathbb{Q}}\left[\int_{T_1}^{T_2} (S_u - F(t,T_1,T_2))du|\mathcal{G}_t\right]$$

Now, we can postulate:

Proposition 3.4.1. General forward price with delivery period. *The forward price of electricity in t with delivery period $[T_1, T_2]$ and settlement against the spot in T_2 is given by*

$$F_{\mathcal{G}}^{\mathbb{Q}}(t,T_1,T_2) = \mathbb{E}^{\mathbb{Q}}\left[\int_{T_1}^{T_2} \frac{1}{T_2 - T_1}\, S_u du|\mathcal{G}_t\right]$$

i.e. the expectation of the weighted integral over the spot.

We remark that the filtration used in the remainder of this chapter will be the historical filtration $\mathcal{F}_t$, i.e. $\mathcal{G}_t = \mathcal{F}_t$. In light of Section 1.2 and the information approach we will then use other filtrations in later parts of this thesis.

3.4.1. Forward Pricing under the Real-world Measure

In order to ease notation we will, as in Benth et al. (2008a), define the following auxiliary functions:

Notation 3.4.1. Auxiliary functions. We define $\bar{\alpha}(t, T_1, T_2)$ and $\bar{\beta}(t, T_1, T_2)$ to be

$$\bar{\alpha}(t, T_1, T_2) = \begin{cases} -\frac{1}{\alpha}\ (e^{-\alpha(T_2-t)} - e^{-\alpha(T_1-t)}) & t \leq T_1 \\ -\frac{1}{\alpha}\ (e^{-\alpha(T_2-t)} - 1) & t \geq T_1 \end{cases}$$

$$\bar{\beta}(t, T_1, T_2) = \begin{cases} -\frac{1}{\beta}\ (e^{-\beta(T_2-t)} - e^{-\beta(T_1-t)}) & t \leq T_1 \\ -\frac{1}{\beta}\ (e^{-\beta(T_2-t)} - 1) & t \geq T_1 \end{cases}$$

Furthermore, let

$$\hat{\alpha}(t, T_1, T_2) = \begin{cases} \frac{1}{\alpha}\ (T_2 - T_1 + \frac{1}{\alpha}\ (e^{-\alpha(T_2-t)} - e^{-\alpha(T_1-t)})) & t \leq T_1 \\ \frac{1}{\alpha}\ (T_2 - t + \frac{1}{\alpha}\ (e^{-\alpha(T_2-t)} - 1)) & t \geq T_1 \end{cases}$$

$$\hat{\beta}(t, T_1, T_2) = \begin{cases} \frac{1}{\beta}\ (T_2 - T_1 + \frac{1}{\beta}\ (e^{-\beta(T_2-t)} - e^{-\beta(T_1-t)})) & t \leq T_1 \\ \frac{1}{\beta}\ (T_2 - t + \frac{1}{\beta}\ (e^{-\beta(T_2-t)} - 1)) & t \geq T_1 \end{cases}$$

Considering the case that $t \leq T_1$ first, we can calculate the forward price under the real-world measure $\mathbb{P}$:

Proposition 3.4.2. Forward price under the real-world measure. *Let $0 \leq t \leq T_1 < T_2$. The forward price in t with delivery in $[T_1, T_2]$ and under the real-world measure $\mathbb{P}$ is then given by*

$$F_{\mathcal{F}}^{\mathbb{P}}(t, T_1, T_2) = \frac{1}{T_2 - T_1} \bigg(\int_{T_1}^{T_2} \Lambda_u du + \bar{\alpha}(t, T_1, T_2) X_t \\ + \bar{\beta}(t, T_1, T_2) Y_t + \psi'_{L_1}(0) \hat{\beta}(t, T_1, T_2) \bigg)$$

Proof. We commence with the basic definition from Proposition 3.4.1

$$F_{\mathcal{F}}^{\mathbb{P}}(t, T_1, T_2) = \mathbb{E}^{\mathbb{P}} \left[\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} (\Lambda_u + X_u + Y_u) du | \mathcal{F}_t \right]$$

Starting with X_t and its definition, we calculate

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \left[\int_{T_1}^{T_2} X_u du | \mathcal{F}_t \right] &= \mathbb{E}^{\mathbb{P}} \left[\int_{T_1}^{T_2} X_t e^{-\alpha(u-t)} du + \int_{T_1}^{T_2} \int_t^u e^{-\alpha(u-s)} dW_s du | \mathcal{F}_t \right] \\ &= X_t\ \bar{\alpha}(t, T_1, T_2) \end{aligned}$$

using the definition of the Itō integral. Similarly,

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\int_{T_1}^{T_2} Y_u du|\mathcal{F}_t\right] &= \mathbb{E}^{\mathbb{P}}\left[\int_{T_1}^{T_2} Y_t e^{-\beta(u-t)} du + \int_{T_1}^{T_2}\int_t^u e^{-\beta(u-s)} dL_s du|\mathcal{F}_t\right] \\
&= \int_{T_1}^{T_2} Y_t e^{-\beta(u-t)} du + \int_{T_1}^{T_2}\int_t^u e^{-\beta(u-s)} \mathbb{E}[L_1] ds\ du \\
&= Y_t\ \bar{\beta}(t,T_1,T_2) + \frac{1}{\beta}\ \mathbb{E}[L_1] \int_{T_1}^{T_2} (1 - e^{-\beta(u-t)}) du \\
&= Y_t\ \bar{\beta}(t,T_1,T_2) + \mathbb{E}[L_1]\hat{\beta}(t,T_1,T_2)
\end{aligned}
$$

Replacing the expectation by the log-moment generating function as in the proof of Proposition 3.2.1 and collecting terms provides the result. □

For t during the delivery period one has the following easy modification of the previous theorem:

Corollary 3.4.1. Forward price during the delivery period. *For $0 \leq T_1 < t \leq T_2$ the forward price is given by*

$$
F_{\mathcal{F}}^{\mathbb{P}}(t,T_1,T_2) = \frac{1}{T_2 - T_1}\ \left(\int_{T_1}^{t} S_u du + (T_2 - t) F_{\mathcal{F}}^{\mathbb{P}}(t,t,T_2)\right)
$$

where the expression on the right hand side is to be calculated according to Proposition 3.4.2.

3.4.2. Forward Pricing under a Risk-neutral Measure

We will now conduct a parametric measure change from the real-world measure to some risk-neutral measure. For the Brownian part of the spot price (i.e. the base component) this corresponds to applying Girsanov's theorem and we refer to Protter (2005, Section III.8) and Shiryaev (1999, Section VII.3b) for technical details. For the Lévy part we will use its equivalent, Esscher transform, and one can find good descriptions in Shiryaev (1999, Section VII.3c), Hubalek and Sgarra (2006), Gerber and Shiu (1994) or Sato (1999, Section 6.33). Both methods can be applied separately as components X_t and Y_t are independent. Remembering the discussion from Section 1.2 every equivalent measure qualifies as pricing measure and thus, a parametric measure change will eventually enable a calibration to observed data (in Chapter 8 with details provided in Section 8.3).

Notation 3.4.2. The pricing measure. We will denote by $\mathbb{Q}_W$ the new measure for the base component and by $\mathbb{Q}_L$ the one of the spike component. Furthermore, $\theta(t)$ will denote a deterministic vector function of measure change parameters, i.e. $\theta(t) = (\theta_W(t), \theta_L(t))^T$. We can aggregate the risk-neutral measure as

$$
\mathbb{Q} = \mathbb{Q}_W \times \mathbb{Q}_L
$$

For the base component of the spot model we define the Radon-Nikodým derivative $Z_W(t)$ as usual

$$Z_W(t) = \frac{d\mathbb{Q}_W}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = \exp\left(\int_0^t \frac{\theta_W(s)}{\sigma}\, dW_s - \frac{1}{2}\int_0^t \frac{\theta_W^2(s)}{\sigma^2}\, ds\right) \tag{3.4}$$

Under the new measure process $\tilde{W}_t$ satisfying

$$d\tilde{W}_t = -\frac{\theta_W(t)}{\sigma}\, dt + dW_t$$

is a Brownian motion. We remark that $\theta_W(t)$ is called the market price of risk (which, in a Black-Scholes world, equals the *Sharpe ratio*). Consequently, under the new measure, the dynamics of X_t are altered as follows

$$dX_t = (\theta_W(t) - \alpha X_t)dt + \sigma d\tilde{W}_t$$

which has solution

$$X_t = e^{-\alpha(t-s)}X_s + \int_s^t e^{-\alpha(t-u)}\theta_W(u)du + \sigma\int_s^t e^{-\alpha(t-u)}d\tilde{W}_t$$

Similarly, for the Lévy process, we define the Esscher transform

$$Z_L(t) = \frac{d\mathbb{Q}_L}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = \exp\left(\int_0^t \theta_L(s)dL_s - \int_0^t \psi_{L_1}(\theta_L(s))ds\right) \tag{3.5}$$

where the parameter function $\theta_L(t)$ is usually called the market price of jump risk. Also, $\psi_{L_1}(u)$ is again the log-moment generating function, i.e. $\psi_{L_1}(u) = \log\mathbb{E}[e^{uL_1}]$ (used as in the proof of Proposition 3.2.1).

L_t remains a Lévy process under the measure $\mathbb{Q}_L$ defined by Equation 3.5. Also, for a constant change of measure (i.e. $\theta_t = \theta$) it has been shown in Lemma 3.3.2 that the integrability condition necessary for an Esscher transform is satisfied for the specific process chosen in Definition 3.3.2.

Benth et al. (2008a, Proposition 4.1) then calculate the price of a forward under the measure $\mathbb{Q}$ in the following way:

Proposition 3.4.3. Forward price under the risk-neutral measure. *Let $0 \le t \le T_1 < T_2$. The forward price with delivery period in $[T_1, T_2]$ under the measure $\mathbb{Q}$ as defined previously is given by*

$$\begin{aligned} F_{\mathcal{F}}^{\mathbb{Q}}(t, T_1, T_2) = \frac{1}{T_2 - T_1}&\left(\int_{T_1}^{T_2} \Lambda_u du + \bar{\alpha}(t, T_1, T_2)X_t + \bar{\beta}(t, T_1, T_2)Y_t\right.\\ &\left.+\int_t^{T_2} \theta_W(s)\bar{\alpha}(s, T_1, T_2)ds + \int_t^{T_2} \psi'_{L_1}(\theta_L(s))\,\bar{\beta}(s, T_1, T_2)ds\right) \end{aligned}$$

Proof. The first three terms are the same as in Proposition 3.4.2. Adjusting for the market price of risk, we calculate

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\int_{T_1}^{T_2}\int_t^u \sigma e^{-\alpha(u-s)} dW_s du \mid \mathcal{F}_t\right] \\
&\qquad = \mathbb{E}^{\mathbb{Q}}\left[\int_{T_1}^{T_2}\int_t^u \sigma e^{-\alpha(u-s)} d\left(\int_0^s \frac{\theta_W(u)}{\sigma}du + \tilde{W}_s\right) du \mid \mathcal{F}_t\right] \\
&\qquad = \mathbb{E}^{\mathbb{Q}}\left[\int_{T_1}^{T_2}\int_t^u e^{-\alpha(u-s)}\theta_W(s) ds du \mid \mathcal{F}_t\right] \\
&\qquad = \int_t^{T_2} \theta_W(s)\bar{\alpha}(s, T_1, T_2) ds
\end{aligned}$$

where in the last step we have applied the stochastic version of Fubini's theorem (cf. Klenke (2006, Section 14.2)) to interchange the order of the integrals. Considering the Lévy part of the result and writing in terms of measure $\mathbb{P}$ the Esscher transform yields

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\int_{T_1}^{T_2}\int_t^u e^{-\beta(u-s)} dL_s du \mid \mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\int_{T_1}^{T_2}\int_t^u e^{-\beta(u-s)} dL_s \frac{Z_L(u)}{Z_L(t)}\, du \mid \mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\int_{T_1}^{T_2}\int_t^u e^{-\beta(u-s)} dL_s \exp\left(\int_t^u \theta_L(s) dL_s - \int_t^u \psi_{L_1}(\theta_L(s)) ds\right) du\right] \\
&= \int_{T_1}^{T_2} \exp\left(-\int_t^u \psi_{L_1}(\theta_L(s))ds\right) \mathbb{E}^{\mathbb{P}}\left[\int_t^u e^{-\beta(u-s)} dL_s \exp\left(\int_t^u \theta_L(s) dL_s\right)\right] du
\end{aligned}$$

We now introduce an artificial differentiation that will help to continue the calculation:

$$\begin{aligned}
\ldots = &\int_{T_1}^{T_2} \exp\left(-\int_t^u \psi_{L_1}(\theta_L(s))ds\right) \\
&\quad \frac{d}{dx}\left(\mathbb{E}^{\mathbb{P}}\left[\exp\left(\int_t^u \left(xe^{-\beta(u-s)} + \theta_L(s)\right) dL_s\right) \mid \mathcal{F}_t\right]\right)\Bigg|_{x=0} du \\
= &\int_{T_1}^{T_2} \exp\left(-\int_t^u \psi_{L_1}(\theta_L(s))ds\right) \\
&\quad \frac{d}{dx}\left(\exp\left(\log\left(\mathbb{E}^{\mathbb{P}}\left[\exp\left(\int_t^u \left(xe^{-\beta(u-s)} + \theta_L(s)\right) dL_s\right) \mid \mathcal{F}_t\right]\right)\right)\right)\Bigg|_{x=0} du
\end{aligned}$$

And now we can continue by inserting the log-moment generating function.

$$
\begin{aligned}
\ldots = &\int_{T_1}^{T_2} \exp\left(-\int_t^u \psi_{L_1}(\theta_L(s))ds\right) \\
&\qquad \frac{d}{dx}\left(\exp\left(\int_t^u \left(\psi_{L_1}\left(xe^{-\beta(u-s)} + \theta_L(s)\right) ds\right)\right)\right)\Big|_{x=0} du \\
= &\int_{T_1}^{T_2} \exp\left(-\int_t^u \psi_{L_1}(\theta_L(s))ds\right)\left(\int_t^u e^{-\beta(u-s)}\psi'_{L_1}\left(xe^{-\beta(u-s)} + \theta_L(s)\right) ds\right. \\
&\qquad \left.\exp\left(\int_t^u \psi_{L_1}\left(xe^{-\beta(u-s)} + \theta_L(s)\right) ds\right)\right)\Big|_{x=0} du \\
= &\int_{T_1}^{T_2} \exp\left(-\int_t^u \psi_{L_1}(\theta_L(s))ds\right) \\
&\qquad \left(\int_t^u e^{-\beta(u-s)}\psi'_{L_1}\left(\theta_L(s)\right) ds \exp\left(\int_t^u \psi_{L_1}\left(\theta_L(s)\right) ds\right)\right) du \\
= &\int_{T_1}^{T_2}\int_t^u e^{-\beta(u-s)}\psi'_{L_1}\left(\theta_L(s)\right) dsdu \\
= &\int_t^{T_2} \psi'_{L_1}\left(\theta_L(s)\right)\bar{\beta}(s, T_1, T_2)ds
\end{aligned}
$$

After collecting terms the proposition is proved. □

As before, we provide a simpler version for the case that t is some time point during the delivery period:

Corollary 3.4.2. Forward price during the delivery period. *For $0 \leq T_1 < t \leq T_2$ the forward price under $\mathbb{Q}$ takes the form*

$$
F_{\mathcal{F}}^{\mathbb{Q}}(t, T_1, T_2) = \frac{1}{T_2 - T_1}\left(\int_t^{T_1} S_u du + (T_2 - t)F_{\mathcal{F}}^{\mathbb{Q}}(t, t, T_2)\right)
$$

where again the last term is calculated according to Proposition 3.4.3.

If the measure change parameter functions are chosen to be constant, i.e. $\theta_L(t) = \theta_L$ and $\theta_W(t) = \theta_W$, the result from Proposition 3.4.3 simplifies.

Corollary 3.4.3. Forward price with constant measure transforms. *With constant market prices of risk the forward price with delivery is given by*

$$
\begin{aligned}
F_{\mathcal{F}}^{\mathbb{Q}}(t, T_1, T_2) = \frac{1}{T_2 - T_1}&\left(\int_{T_1}^{T_2} \Lambda_u du + \bar{\alpha}(t, T_1, T_2)X_t + \bar{\beta}(t, T_1, T_2)Y_t\right. \\
&\qquad \left. + \theta_W\ \hat{\alpha}(t, T_1, T_2) + \psi'_{L_1}\left(\theta_L\right)\hat{\beta}(t, T_1, T_2)\right)
\end{aligned}
$$

Choosing, in particular, both measure change parameters equal to zero we get, not surprisingly, the result of Proposition 3.4.2. Furthermore, we can take limits to reproduce the value of $F_{\mathcal{F}}^{\mathbb{P}}(t, T)$ (see Proposition 3.2.1) as well as find an expression for $F_{\mathcal{F}}^{\mathbb{Q}}(t, T)$:

Corollary 3.4.4. Delivery point forward under pricing measure as limit. *The value of $F_{\mathcal{F}}^{\mathbb{Q}}(t,T)$ is the limit of the result of Proposition 3.4.3 as $T_2 \to T_1$ and given by*

$$F_{\mathcal{F}}^{\mathbb{Q}}(t,T) = \Lambda_T + e^{-\alpha(T-t)}X_t + e^{-\beta(T-t)}Y_t + \int_t^T e^{-\alpha(T-s)}\theta_W(s)ds + \int_t^T e^{-\beta(T-s)}\psi'_{L_1}(\theta_L(s))ds$$

Proof. We have to calculate the limit as the delivery period converges to a delivery point, i.e.

$$\lim_{T_2 \to T_1} F_{\mathcal{F}}^{\mathbb{Q}}(t,T_1,T_2)$$

For each term we encounter a $\frac{0}{0}$ situation and apply L'Hospital's rule. For example

$$\lim_{T_2 \to T_1} \frac{\bar{\alpha}(t,T_1,T_2)X_t}{T_2 - T_1} = \lim_{T_2 \to T_1} \frac{\partial\bar{\alpha}(t,T_1,T_2)X_t}{\partial T_2} = e^{-\alpha(T_1-t)}X_t$$

conducting this calculation for each term and renaming $T_1 = T$ gives the required result. □

3.5. Estimating the Model Parameters

In this section we will discuss methods used to estimate the parameters of our two-factor constant-parameter spot model from observed data. These methods are taken from a variety of papers (as mentioned in Section 3.1). We will use the following notation:

Notation 3.5.1. Observed spot price. We will denote the observed spot price time series by $\hat{S}_t$. We will further assume that time begins in $t = 0$ and ends in $t = T$.

Our spot model consists of two stochastic components and the seasonality function. For purely Gaussian models the most intuitive method to identify the values of the parameters would be to employ the Kalman-Bucy filter (we refer to Bain and Crisan (2008) for the theory of filtering) and this is done, for example in Barlow et al. (2004). Clearly, this method is not suitable for non-Gaussian models. Building on this idea though, some authors propose to estimate all parameters of their spot models simultaneously (see Lucia and Schwartz (2002) or Wilkens and Wimschulte (2007)), for example by applying non-linear least squares methods, but results are not satisfying. Huisman and Mahieu (2003) find that the frequency of jumps is overestimated and generally disentangling the different components does not seem to be working reliably with this approach.

Therefore, we will follow a different approach by trying to estimate the components one by one. Our agenda will be:

1. Filtering out spikes from $\hat{S}_t$
2. Estimating the parameters of the seasonality Λ_t from the remaining time series

3. Estimating the parameters of the base component X_t from the deseasonalised time series without jumps
4. Estimating the parameters of the spike component Y_t

The reason why we commence by filtering (and removing) spikes is because, as large deviations from the "normal" level, they decrease the quality of the fitting in other steps of the procedure.

3.5.1. Filtering Spikes

We will now describe an easy and intuitive algorithm presented in Cartea and Figueroa (2005). The idea is to identify as spikes large deviations from some level considered as "normal".

The recursive algorithm is:

Algorithm 3.5.1. Identifying spikes (recursively)

1. Let $\hat{S}_t^J$ denote the spikes, initialise $\hat{S}_t^J = 0 \; \forall t \leq T$ and fix suitable constant k
2. Identify Λ_t and define the spikeless part of the spot $\hat{S}_t^S = \hat{S}_t - \Lambda_t$
3. Calculate $m = \mathbb{E}[\hat{S}^S]$ and $v = \sqrt{Var(\hat{S}^S)}$
4. For all $0 \leq t \leq T$ check whether $\hat{S}_t^S \geq m + kv$, if so, set $\hat{S}_t^J = \hat{S}_t^S$ and $\hat{S}_t = \Lambda_t$
5. Repeat steps two to five until no more spike is found, $\hat{S}_t^J$ is the spikes series

A discussion of how to identify the parameters of the seasonal component Λ_t is featured in Section 3.5.2. In Borovkova and Permana (2006) and Borovkova et al. (2009) a similar algorithm is used:

Algorithm 3.5.2. Identifying spikes (moving window)

1. Initialise $\hat{S}_t^J = 0 \; \forall t \leq T$, fix some suitable constant k_1, initialise $\hat{S}_t^S = \hat{S}_t$
2. For each t consider the moving window of length k_2, i.e. $S_{t-k_2}, \ldots, S_{t-1}$
3. Calculate m and v for the current window
4. Check whether $\hat{S}_t \geq m + k_1 v$, if so, set $\hat{S}_t^S = (m + k_1 v)$ and $\hat{S}_t^J = \hat{S}_t - \hat{S}_t^S$
5. $\hat{S}_t^J$ is then the collection of (truncated) spikes

Later, we will apply Algorithm 3.5.1 but both algorithms provide equally good results. Obviously, the disadvantage of the first algorithm is that we have to find Λ_t in each iteration. Its advantage is that the whole time series is considered rather than only a window. Generally, there are two problems with this type of algorithm. Firstly, in case a spike is identified, one assumes the value of the Gaussian Ornstein-Uhlenbeck process X_t at that time to be zero or the truncation value, respectively. This problem is negligible. The second problem is more severe. The real goal here is to identify the process Y_t, i.e. a Lévy Ornstein-Uhlenbeck process rather than only the jumps of L_t. We will also have to address the mean-reverting feature of the jump component. Meyer-Brandis and Tankov (2008) propose sophisticated methods to filter out spike components and to estimate their parameters. They are based on *auto-correlation functions* and need a much more difficult machinery. Furthermore, although they are theoretically appealing, they tend to yield non-plausible numbers and, in particular, mean reversion rates in practice. The way in which we will deal with this problem later is not completely consequent academically. Still, we will see that the approach presented here is acceptable and easy yielding plausible results, in particular for mean-reversion rates extracted from real data. We will discuss this further in Section 3.5.4.

3.5.2. The Seasonal Component

The larger part of the movements in electricity prices is due to predictable phenomena (as discussed in Section 1.1). Section 3.3 provided the specific form of Λ_t we will consider

$$\begin{aligned}\Lambda_t = b_0 + b_1 t + b_2\cos(\tfrac{2\pi}{365}t) + b_3\sin(\tfrac{2\pi}{365}t) + b_4\cos(\tfrac{2\pi}{182}t) + b_5\sin(\tfrac{2\pi}{182}t)\\ + b_6\mathbb{1}_{\{t\ mod\ 7=0\}}(t) + b_7\mathbb{1}_{\{t\ mod\ 7=1\}}(t) + \ldots + b_{12}\mathbb{1}_{\{t\ mod\ 7=6\}}(t)\end{aligned}$$

as defined in Definition 3.2.2.

The following short sections will very briefly explain the methods used in the implementations of Section 3.7 and Chapter 8. As mentioned before, we extract Λ_t from the spikeless time series $\hat{S}_t^S$.

3.5.2.1. The Trend Function

The trend function $b_0 + b_1 t$ is the least important component of $\Lambda(t)$ and mainly transforms the time series under consideration into a mean zero series. We obtain parameters b_0, b_1 by a standard linear least squares regression. For most time series the parameters identified like that are very close to zero.

3.5.2.2. Annual Components

The annual component is introduced to capture predictable price alterations due to the season and the weather in particular. Estimating parameters $b_2, \ldots, b_5$ can be done by Fourier analysis. Still, with known frequency (365 days) we can solve explicitly using least squares regression.

The following calculations have been taken from Bloomfield (1976, page 12 et seqq.).

To comply with Bloomfield's notation we let n be the time horizon, $x_t = \hat{S}_t^S - b_0 - b_1 t$ as well as $\omega = \frac{2\pi}{365}$. The least squares setup is

$$\min_{b_2,b_3} T(b_2,b_3) = \sum_{t=0}^{n-1}(x_t - b_2\cos(\omega t) - b_3\sin(\omega t))^2$$

This leads to the derivatives

$$\frac{\partial T}{\partial b_2} = -2\sum_{t=0}^{n-1} \cos(\omega t)(x_t - b_2\cos(\omega t) - b_3\sin(\omega t))$$
$$\frac{\partial T}{\partial b_3} = -2\sum_{t=0}^{n-1} \sin(\omega t)(x_t - b_2\cos(\omega t) - b_3\sin(\omega t))$$

Solving these equations gives

$$\hat{b}_2 = \frac{1}{\Delta}\left(\sum_{t=0}^{n-1} x_t\cos(\omega t)\sum_{t=0}^{n-1}(\sin(\omega t))^2 - \sum_{t=0}^{n-1} x_t\sin(\omega t)\sum_{t=0}^{n-1}\cos(\omega t)\sin(\omega t)\right)$$
$$\hat{b}_3 = \frac{1}{\Delta}\left(\sum_{t=0}^{n-1} x_t\sin(\omega t)\sum_{t=0}^{n-1}(\cos(\omega t))^2 - \sum_{t=0}^{n-1} x_t\cos(\omega t)\sum_{t=0}^{n-1}\cos(\omega t)\sin(\omega t)\right)$$

with auxiliary variable

$$\Delta = \sum_{t=0}^{n-1}(\cos(\omega t))^2\sum_{t=0}^{n-1}(\sin(\omega t))^2 - \left(\sum_{t=0}^{n-1}\cos(\omega t)\sin(\omega t)\right)^2$$

Bloomfield simplifies and approximates these formulae resulting in least squares estimates:

$$\hat{b}_2 = \frac{2}{n}\sum_{t=0}^{n-1} x_t cos(\omega t)\ ,\ \hat{b}_3 = \frac{2}{n}\sum_{t=0}^{n-1} x_t sin(\omega t) \tag{3.6}$$

The overtone suggested by some papers (such as Borovkova and Permana (2006); Borovkova et al. (2009)) can be dealt with equivalently and in turn.

3.5.2.3. The In-week Component

To find estimates for the dummy parameters $b_6, \ldots, b_{12}$ we implement an idea used in Borovkova and Permana (2006). There, the authors calculate the average deviation of each day's price from its week average. The motivation for choosing this approach when compared to, say, another Fourier analysis, is that the weekend vs. workday effect is much better captured. We let $\hat{S}_t^D$ denote the detrended and deseasonalised time series with spikes removed. Furthermore, we let n denote the total number of weeks in the data set. We calculate for $i = 0, \ldots, 6$

$$b_{6+i} = \frac{1}{n}\sum_{j=0}^{n-1}\left(\hat{S}_{7j+i}^D - \frac{1}{7}\sum_{l=1}^{7}\hat{S}_{7j+l}^D\right) \tag{3.7}$$

Then, $b_6, \ldots, b_{12}$ denote the average deviation each of the seven days of the week.

3.5.3. The Gaussian Ornstein-Uhlenbeck Process

Let the random residuals be $\hat{S}_t^R = \hat{S}_t^S - \Lambda_t$. We want to identify $\hat{S}_t^R$ with the base component X_t. The Gaussian Ornstein-Uhlenbeck process is *AR(1)*, an autoregressive series of order one. As such, one way to extract the parameters is by solving the so-called *Yule-Walker* equations. We refer to Carmona (2004, page 268 et seqq.) for the details. As the order is one, we can also apply an easier method. Rewriting Equation 3.1 gives

$$X_t = e^{-\alpha} X_{t-1} + \sigma \int_{t-1}^{t} e^{\alpha(t-s)} dW_s$$

By Itō's isometry the conditional variance of X_t is

$$s^2 = Var(X_t \mid \mathcal{F}_{t-1}) = \int_{t-1}^{t} \sigma^2 e^{2\alpha(t-s)} ds = \frac{\sigma^2}{2\alpha}\left(1 - e^{2\alpha}\right)$$

Setting $m = e^{\alpha}$ we can write $X_t = mX_{t-1} + s\epsilon_t$ where ϵ_t is standard normal. Thus, estimators for m and s are provided by a simple least squares regression from $\hat{S}_{t-1}^R$ to $\hat{S}_t^R$. We can then recover estimates of our parameters by solving

$$\alpha = -\log(m) \ , \ \sigma = s\sqrt{\tfrac{1-m^2}{-2log(m)}}$$

Of course we can only treat $\hat{S}_t^R$ as our base component X_t if it satisfies the desired properties, in particular the one of (weak) *stationarity*. We will later apply the Dickey-Fuller statistical test to check for stationarity.

3.5.4. The Lévy Ornstein-Uhlenbeck Process

We want to find estimates for the parameters of the spike component of our spot model. These are: $\beta, \lambda, p, q, \eta_1, \eta_2$. Remember that the time series $\hat{S}_t^J$ contains the spikes filtered out in Section 3.5.1.

3.5.4.1. Jump Size Parameters

We remarked in Section 3.5.1 that time series $\hat{S}_t^J$ is, strictly speaking, not Y_t. On the one hand, Y_t is a mean-reverting process and thus it will feature after each jump parts that are too small to be filtered out by our method. On the other hand, some non-zero entries in $\hat{S}_t^J$ will really belong to a bigger jump in the vicinity. Thus, as in Metka (2008), we will only consider so-called proper jumps from now on:

Notation 3.5.2. Proper jumps. We will call a positive jump in $\hat{S}_t^J$ proper if $\hat{S}_t^J \neq 0$ and if $\hat{S}_t^J \geq \hat{S}_{t-1}^J$. We will call a negative jump proper if $\hat{S}_t^J \neq 0$ and if $\hat{S}_t^J \leq \hat{S}_{t-1}^J$.

It is now easy to obtain estimators for p, q, η_1, η_2 and λ. The frequency of proper jumps provides the estimator for λ. Similarly, p and q are identified by looking at the sign of proper jumps. The exponential density is $\eta_1 e^{-\eta_1 x}$ and expectation $\frac{1}{\eta_1}$ and thus taking the inverse of the mean of the sample gives the estimators for η_1 and η_2, respectively.

3.5.4.2. Mean-reversion Rate

Dealing with the mean reversion parameter β is more difficult, as announced in Section 3.5.1. Meyer-Brandis and Tankov (2008) calculate the empirical auto-correlation function γ of $\hat{S}_t - \Lambda_t$. They regress that function against the sum of two exponentials, i.e.

$$\gamma_{\hat{S}-\Lambda}(h) = w_1 e^{-\frac{h}{\alpha_1}} + w_2 e^{-\frac{h}{\alpha_2}}$$

For annualised EEX data from 2000 until 2007 they find α_1 around 3 to 5 and α_2 around 100. The authors conjecture that $\beta = \alpha_1$, the mean-reversion rate of Y_t whereas $\alpha = \alpha_2$ is the one of the Gaussian component.

We remark that this method can easily be adapted to more than two components. The regression can be justified by calculating the analytical expression for the auto-correlation function of $X + Y$, assuming for the moment strong stationarity (we remark that Ornstein-Uhlenbeck processes are only weakly stationary). This result is taken from Cont and Tankov (2004, page 484) and exactly has the form of the above regression:

Theorem 3.5.1. Analytical auto-correlation function. *Assuming strong stationarity for both X_t and Y_t, the analytical auto-correlation function of X_t+Y_t and $X_{t+s}+Y_{t+s}$ is*

$$\gamma_{X+Y}(t, t+s) = \frac{e^{-\alpha s} Var(X_t) + e^{-\beta s} Var(Y_t)}{Var(X_t) + Var(Y_t)}$$

where $t < s$.

Proof. If Y is strongly stationary, we get that $\mathbb{E}[Y_t] = \frac{1}{\beta}\,\mathbb{E}[L_1]$ (this is as in the proof of Proposition 3.2.1). Thus, we can calculate the auto-covariance function of Y

$$\begin{aligned} Cov(Y_t, Y_{t+s}) &= \mathbb{E}[Y_t Y_{t+s}] - \mathbb{E}[Y_t]\mathbb{E}[Y_{t+s}] \\ &= \mathbb{E}[Y_t \mathbb{E}[Y_{t+s}|\mathcal{F}_t]] - \mathbb{E}[Y_t]^2 \\ &= e^{-\beta s}\mathbb{E}[Y_t^2] + \tfrac{1-e^{-\beta s}}{\beta}\mathbb{E}[L_1]\mathbb{E}[Y_t] - \mathbb{E}[Y_t]^2 \\ &= e^{-\beta s} Var(Y_t) \end{aligned}$$

For $X + Y$ we use independence and the definition

$$\gamma_{X+Y}(t, t+s) = \frac{Cov(Y_t, Y_{t+s}) + Cov(X_t, X_{t+s})}{\sqrt{Var(Y_t)Var(X_t) + Var(Y_{t+s})Var(X_{t+s})}}$$

of the auto-correlation function to arrive at the required result. □

Although this regression approach is well justified, it turns out that it yields unrealistic estimates for the value of β. In the empirical section of their paper, Meyer-Brandis and Tankov choose (quite arbitrarily) different values from those identified by the regression. Also, in Metka (2008) (Master thesis that provided the empirical results of Benth et al. (2008a)) the author finds $\alpha > \beta$ using a variant of the method in which the value of α is identified as in Section 3.5.3. Clearly, we expect spikes to mean-revert faster than the base component.

Thus, we propose an extremely simple method that nonetheless works fine.

Algorithm 3.5.3. Estimating spike mean-reversion

1. Fix constant k_β and set $v = \sqrt{Var(\hat{S} - \Lambda)}$
2. For each time of a proper jump t_i find smallest j_i with $| \hat{S}_{t_i+j_i} - \Lambda_{t_i+j_i} | < vk_\beta$
3. Add all j_i and divide the result by the number of proper jumps
4. Take the inverse of that number as an estimator for β

The idea is to count the days that it takes for the spot to return to some "normal" level after each proper jump. We then add these lengths and divide the result by the total number of proper jumps. The inverse of that average number can be used as an estimator for β. Again, the "normal" level is specified by using the empirical standard deviation multiplied by a suitable constant k_β.

3.6. Simulating the Model

In this section we will briefly discuss how to simulate price paths for the specific model chosen in Section 3.3. As the base component is obvious, we will concentrate on how to simulate Y_t. This task is divided into simulating the jump sizes and the jump times. For the jump size one draws a binomial random variable with parameter p to determine whether the jump will be a positive or a negative one. Then, an exponentially distributed variable is drawn with parameter η_1 or η_2. Denote by D_{t_i} the jump size of the ith jump.

Cont and Tankov (2004, page 174) describe the easiest way to simulate a compound Poisson process from $t = 0$ to some T.

Algorithm 3.6.1. Compound Poisson / Lévy Ornstein-Uhlenbeck simulation

1. Initialise $i = 1$
2. Draw an exponentially distributed random variable $t_i \sim exp(\lambda)$
3. Find jump size D_{t_i} as above
4. If $\sum_{j=1}^{i} t_j < T$, increase $i = i + 1$ and repeat from step 2, else proceed
5. Calculate $N_t = \sup\{i : \sum_{j=1}^{i} t_j \leq t\}$
6. The compound Poisson in t is then $\sum_{i=1}^{N_t} D_i$, whereas $Y_t = \sum_{i=1}^{N_t} D_i \; e^{-\beta(t-t_i)}$

Still, for this algorithm a lot of random variables have to be drawn and it is not very efficient. An improved version, as presented in Cont and Tankov (2004), makes use of the following results from the theory of Poisson processes and uniformly distributed random variables (cf. Cont and Tankov (2004, page 45 et seqq.)):

Definition 3.6.1. Dirichlet distribution. Let $U_1, \ldots, U_n$ be independent uniformly distributed random variables on $[a, b]$. Let $V_1, \ldots, V_n$ be the corresponding order statistics. Then

$$D_n([a,b]) = \frac{n!}{(b-a)^n} \mathbb{1}_{a<v_1<\ldots<v_n<b}(x)$$

is the density of the V_i and called Dirichlet distribution.

Lemma 3.6.1. Law of sum of exponential random variables. *Given n exponentially distributed random variables τ_i $(1 \leq i \leq n)$ and their sums $T_j = \tau_1 + \ldots + \tau_j$ the law of $(T_1, \ldots, T_n)$ knowing $T_{n+1} = T$ is $D_n([0,T])$.*

This means that we can simulate exponentially distributed random times using order statistics of uniformly distributed random variables. We only need to know the right end of the interval, in our case the length of the simulation T. As the number of jumps of a compound Poisson process in interval $[0, T]$ is Poisson distributed with parameter λT, we can now state (cf. Cont and Tankov (2004)):

Algorithm 3.6.2. Lévy Ornstein-Uhlenbeck simulation. More efficient algorithm.

1. Draw one Poisson variable N with parameter λT
2. Simulate N independent, uniformly distributed random variables on $[0, T]$
3. Calculate N jump sizes D_i for $0 < i \leq N$
4. Return $Y_t = \sum_{i=1}^{N} \mathbb{1}_{U_i<t} D_i e^{-\beta(t-U_i)}$

3.7. Empirical Study

3.7.1. EEX Spot from 2002 until 2009

In this section we will apply the methods from Section 3.5, illustrate and discuss briefly. We will consider the day-ahead baseload price from 01/01/2002 until 31/12/2008. The data set thus consists of seven years and 2556 data points. The graph of the price is illustrated in Figure 3.7.1. It can be observed that during the years 2006, 2007 and 2008 the market exprienced a time of turbulence. Our model as described in Section 3.3 has constant parameters. Thus, we might expect rather problematic fitting for these years. In later chapters, such as Chapter 8, much shorter samples will be considered.

Generally, some of the properties discussed in section Section 1.1 can be identified: a slow trend, extreme positive spikes (in a range up to 300 €) and high volatility.

The spike filtering Algorithm 3.5.1 was applied to the data set with constant $k = 2.6$. The result is illustrated in Figure 3.7.2. The figure shows two phenomena: for the period under consideration there are much more positive than negative spikes. Also, the figure confirms the finding of many authors such as Meyer-Brandis and Tankov (2008) or Borovkova and Permana (2006) that spikes seem to happen in clusters. We will see a different type of behaviour in Chapter 8 for more recent

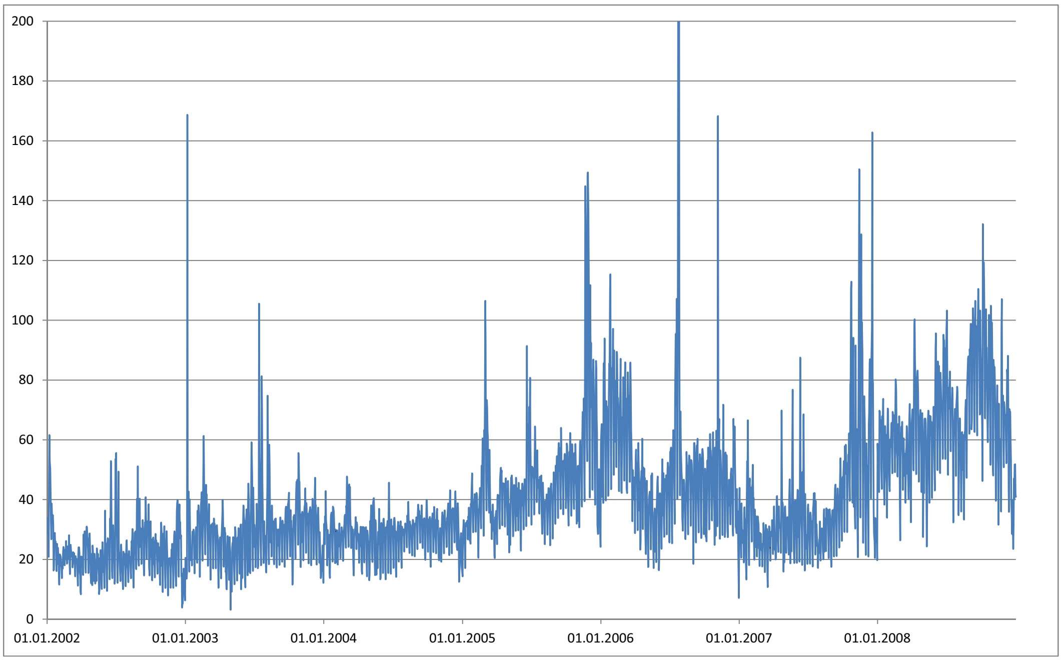


Figure 3.7.1.: EEX spot price 2002-2009. Day-ahead EEX baseload price from 01/01/2002 until 31/12/2008. Cut off at 200 €.

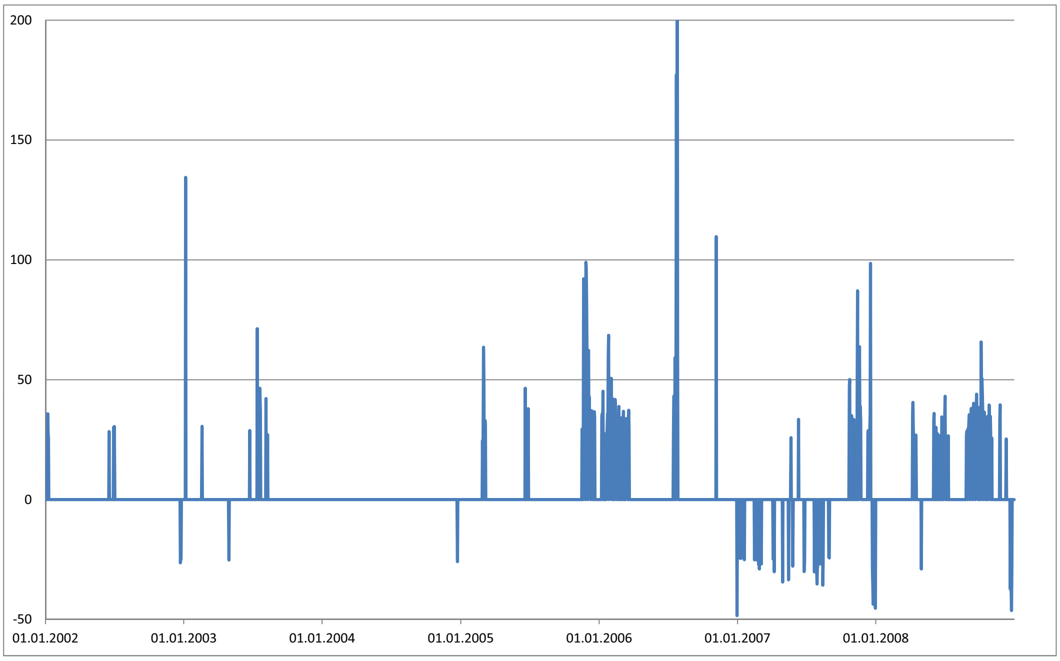


Figure 3.7.2.: EEX spot price 2002-2009: Filtered spikes. Time series $\hat{S}_t^J$

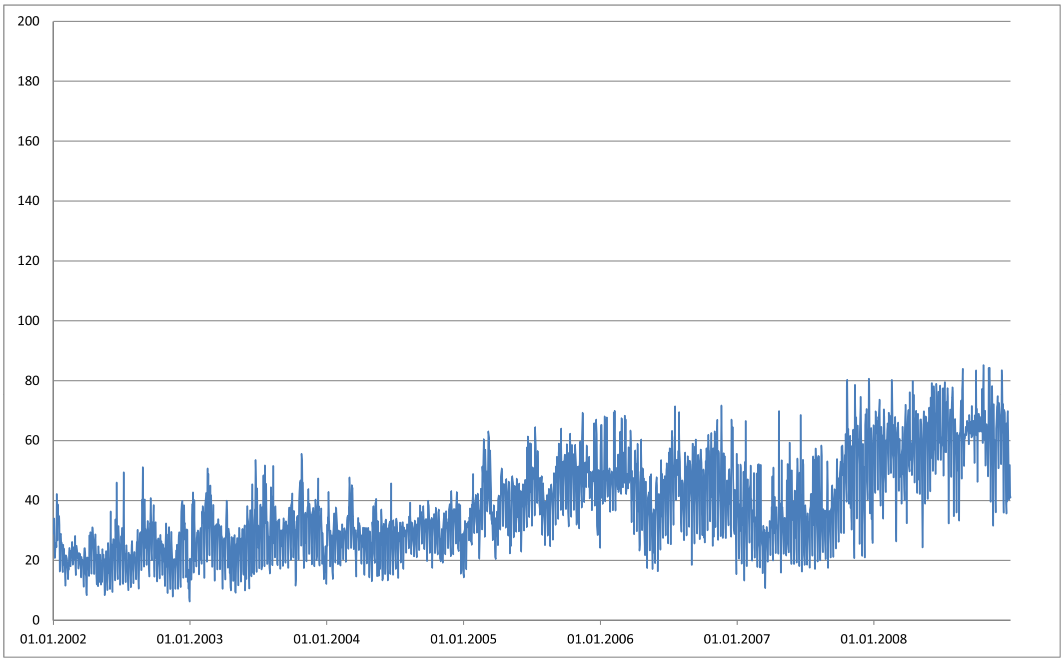


Figure 3.7.3.: EEX spot price 2002-2009: Without spikes. Time series $\hat{S}_t^S$.

data. In Figure 3.7.3 it can be observed that the range of prices has been reduced dramatically.

The time series without spikes $\hat{S}_t^S$ was then deseasonalised and resulting estimates are given in Table 3.7.2. Figure 3.7.4 illustrates the linear drift component whereas Figure 3.7.5 is the remaining time series.

Figure 3.7.6 shows the result of capturing yearly effects in the price as described in Section 3.5.2.2. The overtone hardly has any effect on the shape of the seasonal function. Other data sets will show a more pronounced half-yearly behaviour. We remark that in Figure 3.7.6 the week average of the residual $\hat{S}_t^S - b_0 - b_1 t$ is plotted for clarity. Generally, it is astonishing that the seasonal effect is relatively small.

Figure 3.7.7 shows the estimated dummy variables for the weekdays. The graph matches our intuition as exactly two days of the week feature a systematic negative deviation (the weekend) whereas the other five days have higher prices (workdays). We can even deduce from the shape of the graph that 01/01/2002 was a Tuesday. Figure 3.7.8 shows the remaining time series $\hat{S}_t^R$.

The two Gaussian parameters α and σ are fitted following the linear least squares approach from section Section 3.5.3 and resulting parameter estimates are provided in Table 3.7.2.

We can check for stationarity of the time series $\hat{S}_t^R$ by applying the Dickey-Fuller statistical test. This test checks hypothesis H_0 that the time series under consideration is not stationary. We find a test statistic of $DF(\hat{S}_t^R) = -13.851$ which is smaller than the corresponding $1\%, 5\%$ and 10% levels $(-2.58,\ -1.95,\ -1.62)$ and thus stationarity cannot be rejected. Testing for normality of the residuals (for example with Kolmogorow-Smirnow) or for white noise (Ljung-Box) does not yield

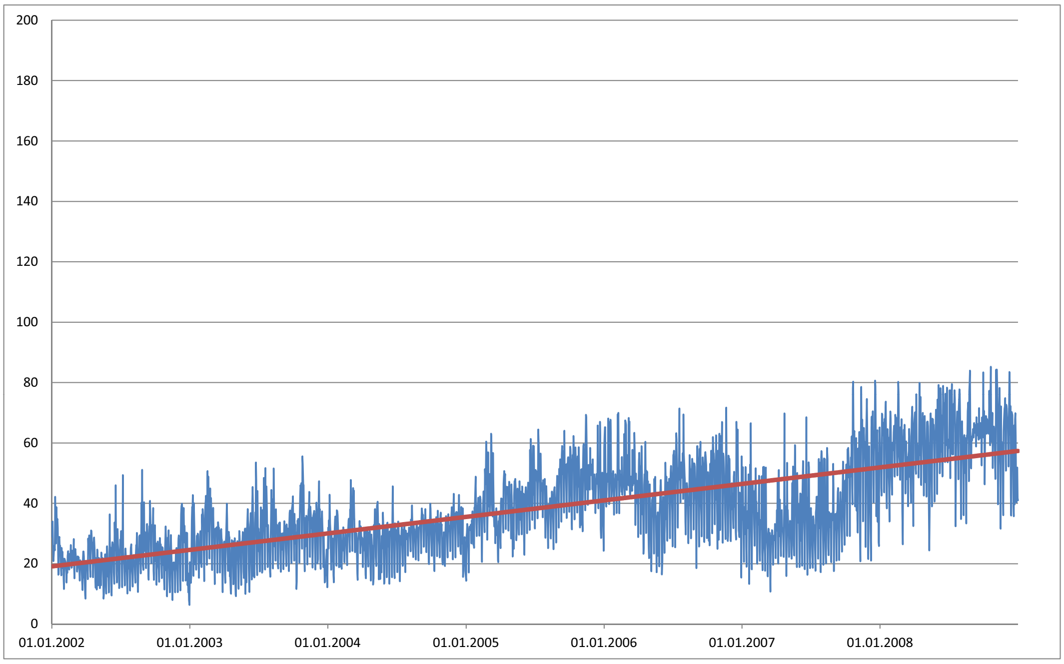


Figure 3.7.4.: EEX spot price 2002-2009: Linear trend. Time series $\hat{S}_t^S$.

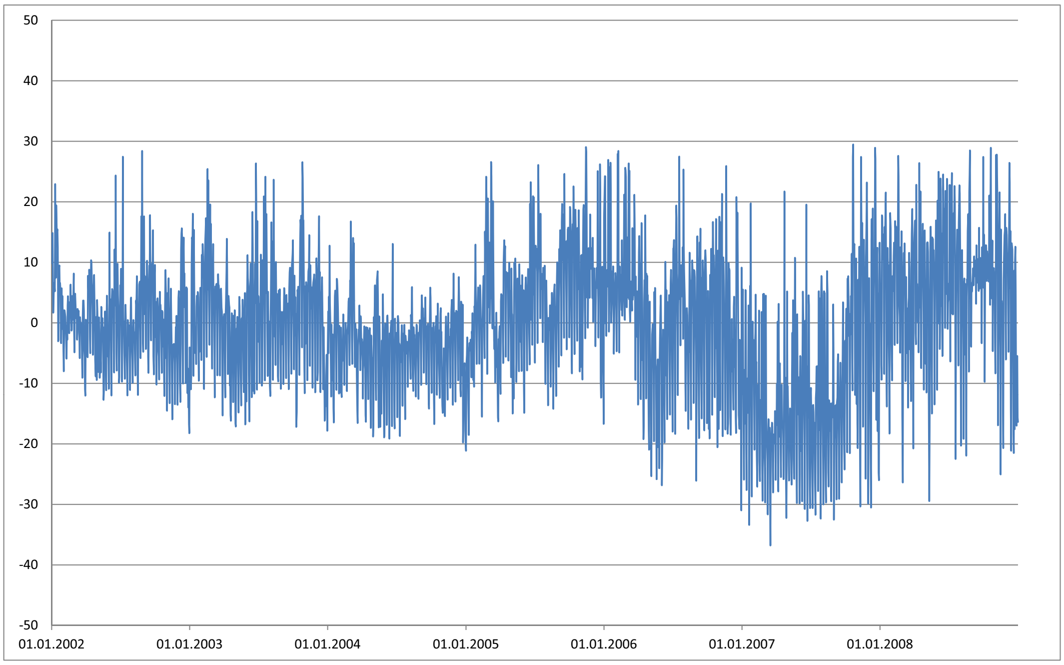


Figure 3.7.5.: EEX spot price 2002-2009: Without trend. Time series $\hat{S}_t^S - b_0 - b_1 t$.

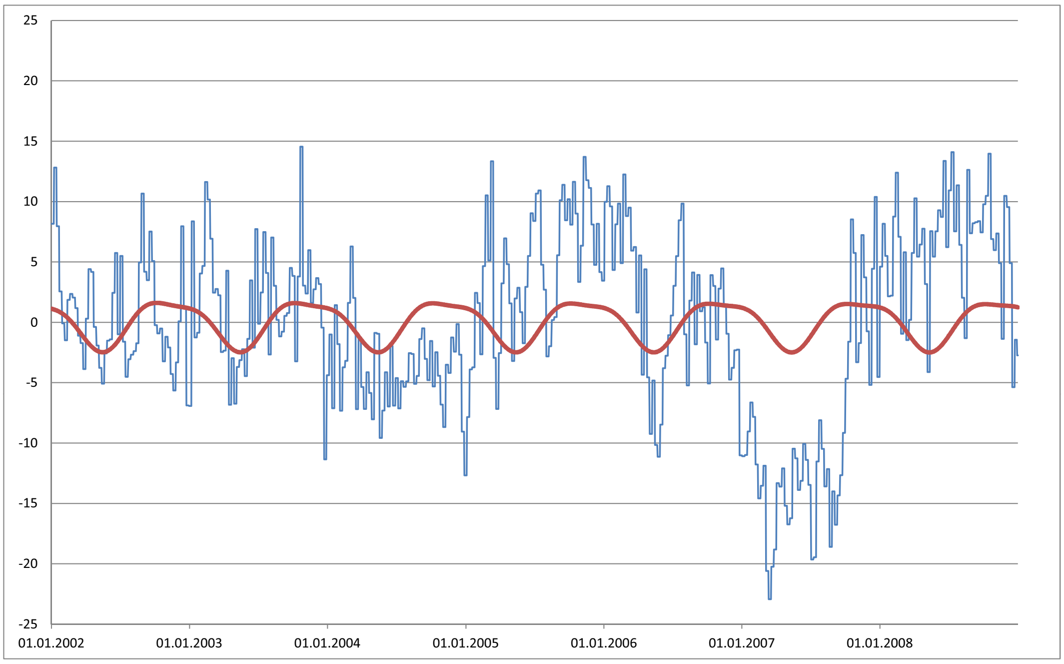


Figure 3.7.6.: EEX spot price 2002-2009: Seasonality. Plotted against the weekly average of $\hat{S}_t^S - b_0 - b_1 t$ for clarity.

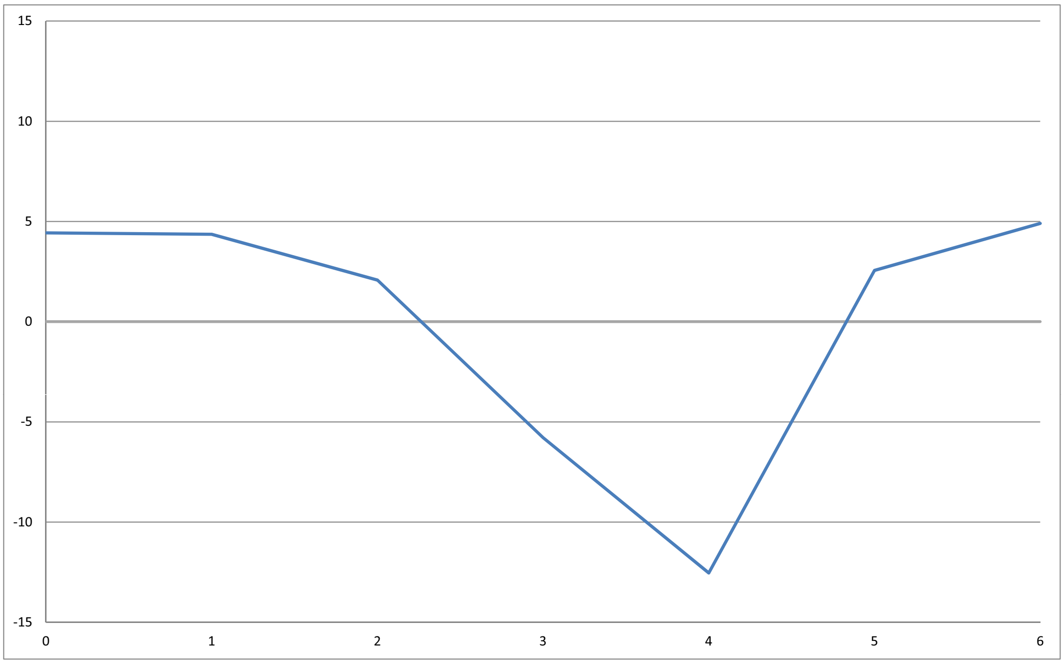


Figure 3.7.7.: EEX spot price 2002-2009: Weekday effect. Dummy variables to be added to the different days of the week.

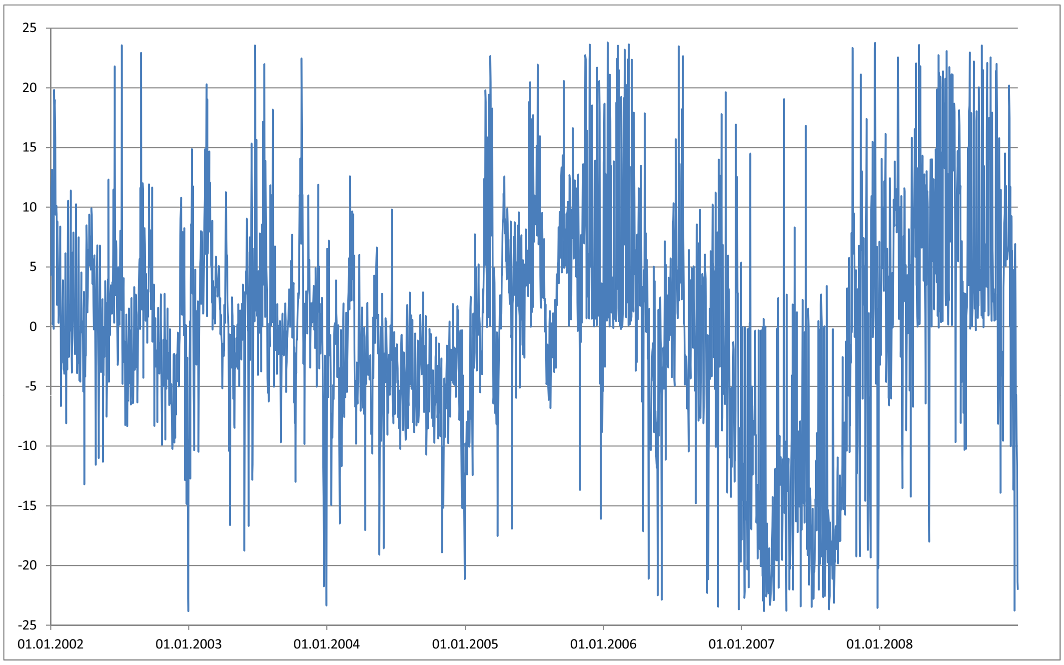


Figure 3.7.8.: EEX spot price 2002-2009: Without weekday effect. Time series $\hat{S}_t^R$.

Mean	Stnd. Dev.	Skewness	Kurtosis
-0.0046	9.11670	-0.0817	0.3041

Table 3.7.1.: EEX spot price 02-09: Moments of Gaussian component. First four moments of $\hat{S}_t^R$.

such satifying results. Still, test statistics are not far off the significance levels and improve when considering shorter data sets.

Moreover, we can check whether the Gaussian Ornstein-Uhlenbeck process is suited for $\hat{S}_t^R$ by looking at the moments provided in Table 3.7.1. Skewness and kurtosis are close to zero.

Estimated parameters of spikes are again given in Table 3.7.2. We remark that, in particular, mean-reversion rates have meaningful values when compared to each other. A β of around 0.9 means that we will expect a spike to last one day on average whereas the base component mean reverts in two to three days. Overall, one has around 7% spikes divided into positive and negative ones at a ratio 3 : 1 with positive spikes being larger than negative ones as $\frac{1}{\eta_1} = 41.84 > 30.30 = \frac{1}{\eta_2}$. This is typical of the time period under consideration.

A comparison between the moments of the original time series $\hat{S}_t$ and 5000 simulated paths is given in Table 3.7.3. The model is able to reproduce moments very well. To check robustness of the parameters, Table 3.7.2 also shows the variance of estimated parameters from 3000 simulations of the model. Here, we experience a high degree of robustness for most parameters, in particular the important ones

Param.	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
Est.	18.86	0.015	1.284	-1.32	-0.12	0.655	4.706	4.488	4.364	2.190
Var.	0.854	0.001	0.612	0.599	0.606	0.598	0.399	0.401	0.398	0.390
Param.	b_{10}	b_{11}	b_{12}	α	σ	λ	p	η_1	η_2	β
Est.	-5.37	-12.46	2.080	0.416	8.325	0.067	0.763	0.024	0.033	0.920
Var.	0.407	0.392	0.407	0.035	0.232	0.004	0.044	0.001	0.003	0.025

Table 3.7.2.: EEX spot price 02-09: Estimated parameters and variance. From EEX spot price and variance of 3000 simulations of the spot.

	Mean	Stnd. Dev.	Skewness	Kurtosis
Orig.	40.1576	21.7497	2.1647	12.2400
Sim.	40.6005	21.9550	1.7706	15.3769

Table 3.7.3.: EEX spot price 02-09: Comparison of moments. First four moments from the original data set $\hat{S}_t$ and from 5000 simulations with estimated parameters.

of the stochastic processes X_t and Y_t (for the trigonometric functions a switching between sinus and cosinus explains the lack of robustness).

The final shape of the process Y_t with estimated parameters is illustrated in Figure 3.7.9. The figure has a smaller time period in order to better depict the mean-reversion effects after a jump. Figure 3.7.10 shows one example of a simulated price path. It has already been mentioned that the years 2007 and 2008 are difficult because 2007 featured a long period of low prices with 2008 witnessing higher prices. This behaviour is not observable in the simulation due to our choice of constant parameters. Generally speaking, one will get a more realistic picture for shorter data sets. We also have negative simulated prices. Their number is small and, as discussed in Section 1.1, this is not entirely unrealistic for the underlying electricity. In recent years the market has exhibited a growing number of negative prices and some papers even discuss this matter from a modelling perspective (as for example Schneider (2011)).

3.7.2. EEX Spot from 01/2002 until 05/2004

Here, we want to address two aspects mentioned in the previous section. We will consider the same EEX data used by Borovkova and Permana (2006). That is: 01/01/2002 until 13/05/2004. Firstly, we show in Figure 3.7.11 that for this period the overtone of the seasonality function is much more pronounced than before. Secondly, we concluded that the data set from Section 3.7.1 might be too long for a spot model with constant parameters which will be confirmed by better fitting results for this data set. The observed spot in Figure 3.7.12 and the corresponding simulated path of Figure 3.7.13 illustrate this point. Graphically, the shorter period provides better fitting results. Statistically, the Dickey-Fuller test once again confirms stationarity of $\hat{S}_t^R$ (with $DF(\hat{S}_t^R) = -9.68 < -2.58 < -1.95 < -1.62$, i.e. $1\%, 5\%$ and 10% levels). Moreover, we test $\hat{S}_t^R$ for normality and the Kolmogorow-Smirnow statistic is $KS(\hat{S}_t^R) = 1.33$ so that for this data set H_0 (normality) is indeed accepted

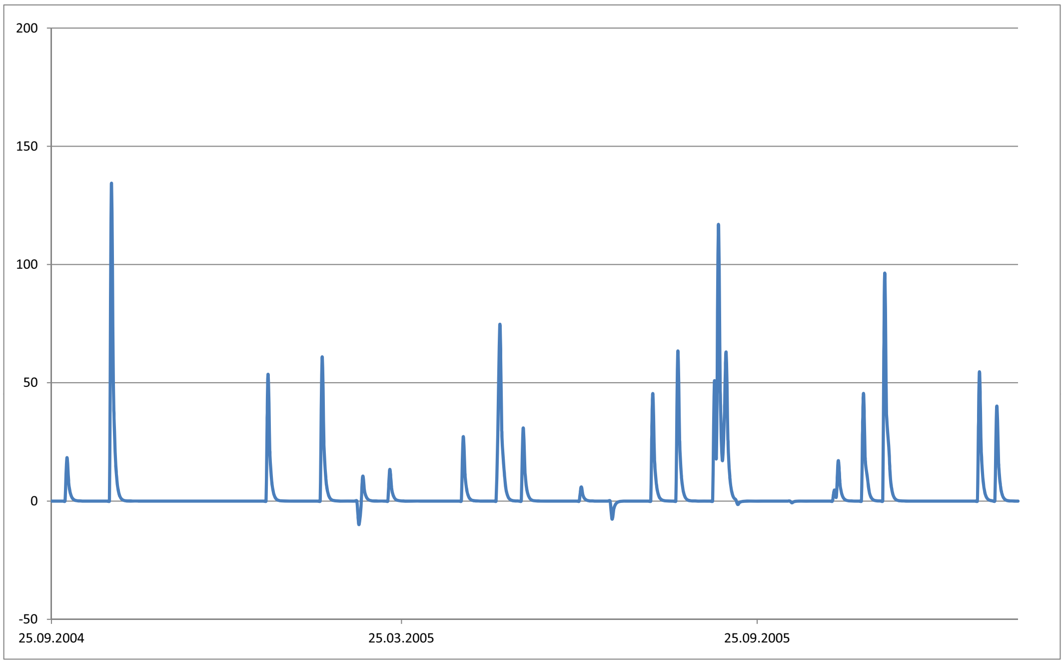


Figure 3.7.9.: EEX spot price 2002-2009: Simulated spikes. For a shorter time period for clarity.

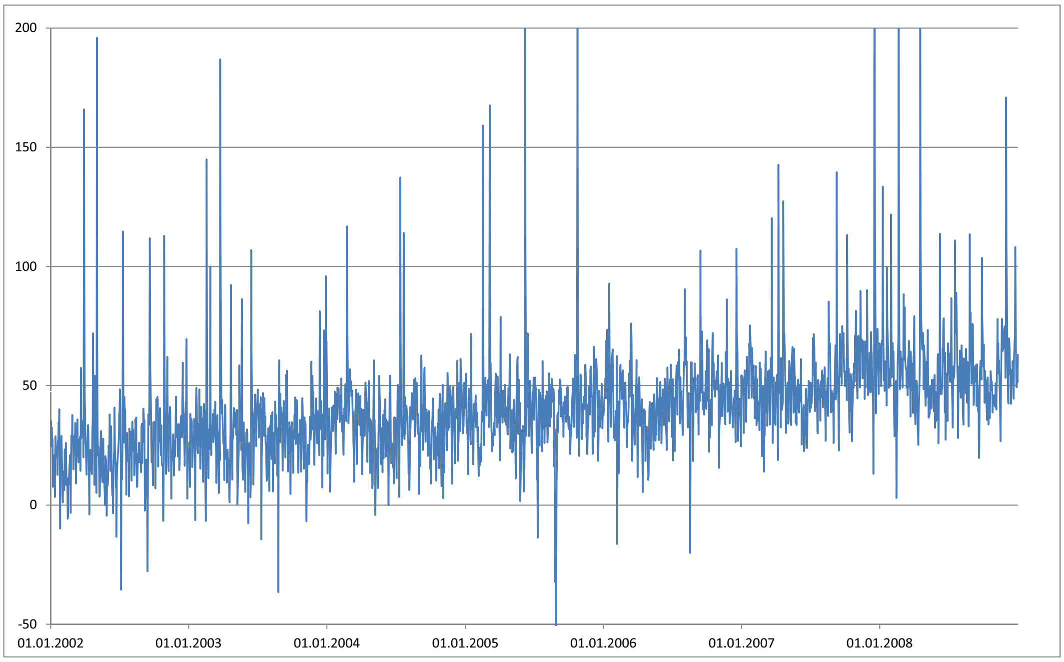


Figure 3.7.10.: EEX spot price 2002-2009: Simulated spot price. Over the whole time period.

	Mean	Stnd. Dev.	Skewness	Kurtosis
Orig.	26.5664	10.2492	1.5245	12.2527
Sim.	26.3923	11.0842	1.4547	17.1748

Table 3.7.4.: EEX spot price 01/02-05/04: Comparison of moments. First four moments from the original data set $\hat{S}_t$ and from 5000 simulations with estimated parameters.

Param.	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
Est.	20.31	0.012	1.077	0.837	0.420	1.260	2.973	3.980	3.829	1.250
Var.	0.585	0.001	0.404	0.421	0.412	0.420	0.331	0.338	0.340	0.338
Param.	b_{10}	b_{11}	b_{12}	α	σ	λ	p	η_1	η_2	β
Est.	-4.707	-9.862	2.538	0.585	4.359	0.079	0.609	0.046	0.064	0.932
Var.	0.338	0.334	0.337	0.072	0.206	0.008	0.073	0.005	0.007	0.036

Table 3.7.5.: EEX spot price 01/02-05/04: Estimated parameters and variance. From EEX spot price and variance of 5000 simulations of the spot.

at the 5% level (with critical values $1.22 < KS(\hat{S}_t^R) < 1.36 < 1.63$).

Table 3.7.4 shows the original and simulated moments of 5000 simulations. Table 3.7.5 gives parameter estimates and their robustness, based on variances of, again, 5000 simulation. Furthermore, interpreting spike parameters η_1 and p confirms that positive jumps are larger and more probable than negative ones. This is the typical price behaviour for the earlier years of the EEX. As mentioned before, we will encounter another situation for more recent data in Section 8.4.2.

A further analysis for this data set is left out for brevity. Two more market scenarios and their corresponding calibration procedures will be discussed in less detail in Chapter 8.

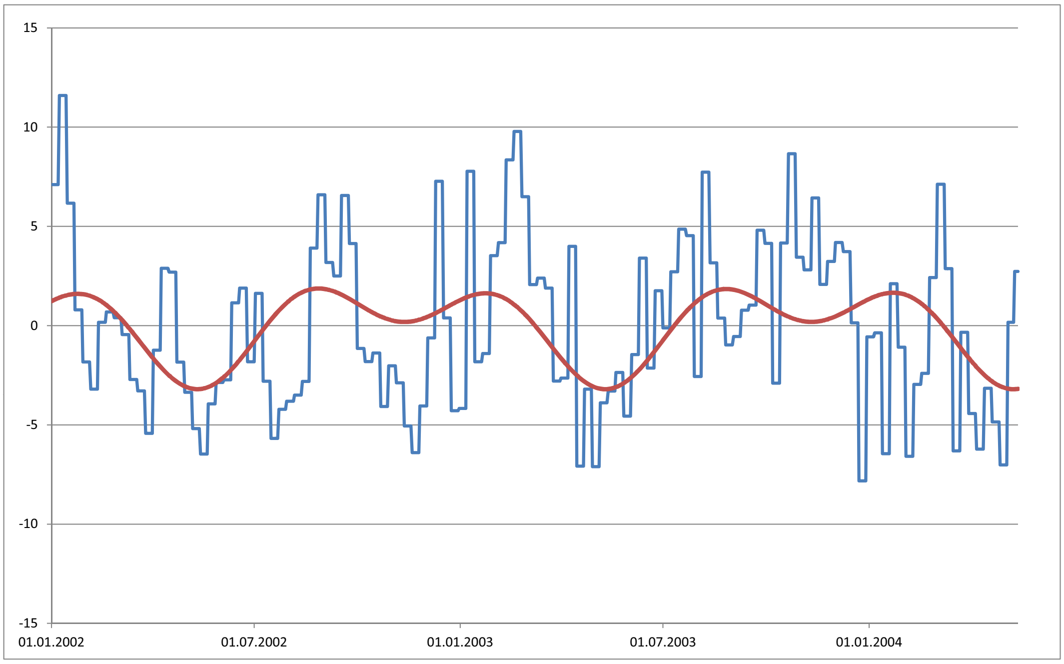


Figure 3.7.11.: EEX spot price 01/02-05/04: Seasonality function. With pronounced half-year effect.

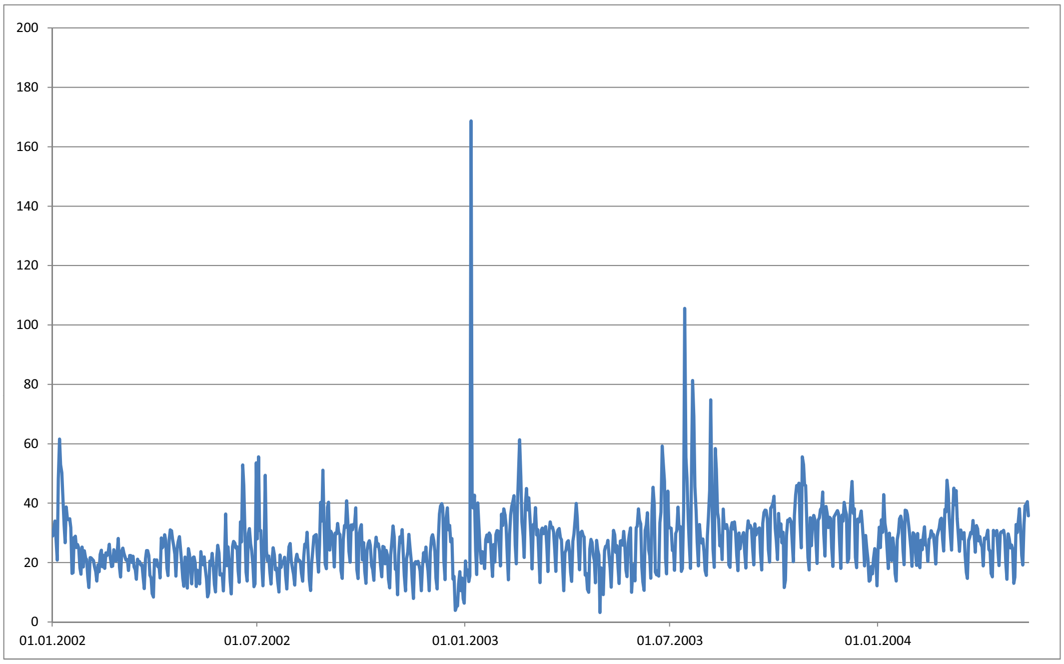


Figure 3.7.12.: EEX spot price 01/02-05/04. Daily day-ahead baseload price.

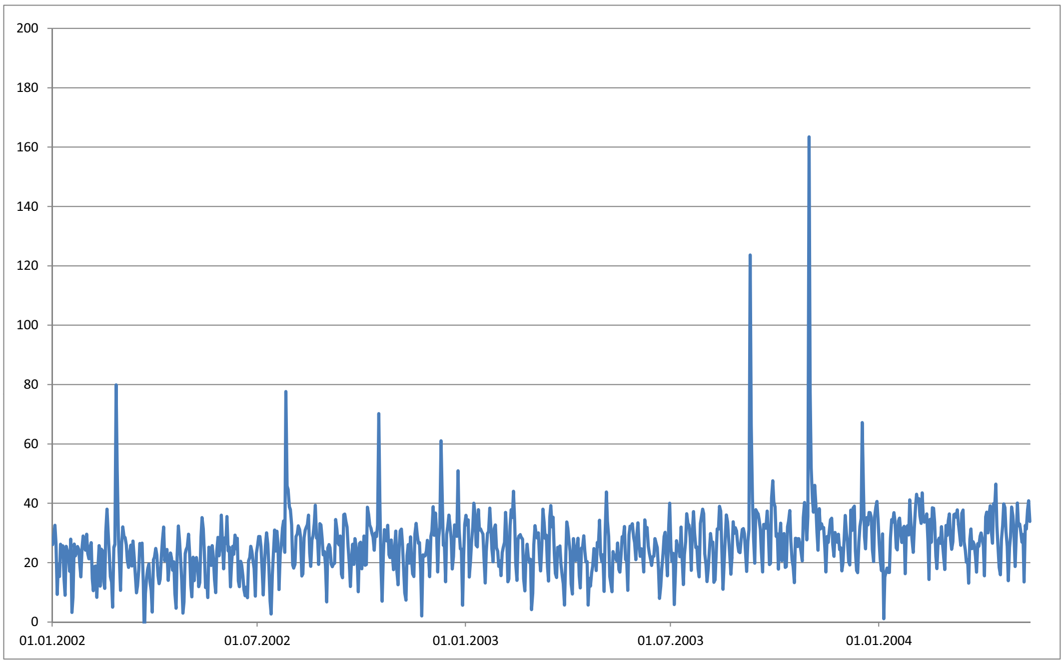


Figure 3.7.13.: EEX spot price 01/02-05/04: Simulated spot price. Over the time period.

Chapter 4.

Risk Premium and Information Approach

4.1. Literature Overview and Summary

The difference between forward and expected spot prices is usually called the *risk premium* and will be formally introduced in Section 4.2. There is a long tradition of research in this area and we will carefully introduce and define the important objects and their relations in the first part of this chapter. Furthermore, the risk premium is also the starting point for the information approach, i.e. the subject of this thesis (cf. Section 1.2) as well as of the initiating paper Benth and Meyer-Brandis (2009). We will briefly summarise and put into context this paper in the second part of this chapter.

The risk premium has been detected for a variety of commodities and financial assets. On classical financial markets the risk premium is found for stocks and stock indices in MacKinlay and Ramaswamy (1988, US market) and Yadav and Pope (1993, UK market). Fama (1984) analyses its properties for exchange rates. For commodities Fama and French (1987) examine prices for metals, crops and other commodities. Wei and Zhu (2006) find economically significant risk premia in the US gas market.

Generally, the significance of the risk premium can be assessed quoting Cochrane (2005, page 451):

> We have to get used to the fact that most returns and price variation come from the variation in risk premia, not variation in expected cash flows, interest rates, etc.

On electricity markets, empirical research has shown a rather inconclusive and random behaviour of the risk premium. For example, Longstaff and Wang (2004) prove that the risk premium exists and is significant and positive on average for high-frequency data of the *PJM* (the Pennsylvania-New-Jersey-Maryland) market. They also find that the risk premium is correlated negatively with price volatility and positively with spot skewness (confirming Bessembinder and Lemmon (2002)). Furthermore, Lucia and Schwartz (2002) examine short-term futures (with a delivery period of one week) traded on the Scandinavian *NordPool*. They, too, find a statistically positive premium that depends in particular on the season during which the contract matures, being highest in winter and zero in summer. For the Spanish electricity market and forwards with maturity within two months, Furiò and Meneu (2010) find that the risk premium decreases with unexpected variations in demand but increases in unexpected variations of the level of hydroelectric energy capacity. Moreover, Diko et al. (2006) find a term structure for the risk premium for data from the German, French and Dutch markets that features a change of sign and negative values for large time to maturity. Their results are similar to Kolos and Ronn (2008), who use EEX and PJM data and include oil and gas as more mature markets for comparison. A link between gas storage and electricity forwards is established in Douglas and Popova (2008) in terms of the moments of the electricity spot price distribution confirming again the analysis of Bessembinder and Lemmon (2002).

It is noticeable that incomprehension prevails as to the true character of the risk premium. From a modelling perspective, Bessembinder and Lemmon (2002) present a very influential one-period model in which the risk premium depends on the variance and the skewness of the spot price. Their model features retailers' demand as the only exogenous variable and they deduce the risk premium by applying market clearing and equilibrium arguments. Benth et al. (2008a) try to explain the term structure of the risk premium taking market power and risk aversion of retailers and producers into consideration (this approach will be discussed in more detail in Chapter 7). They succeed in explaining the positive premium for short-term contracts, the change of sign and the negative premium for large time to maturity by different hedging attitudes and investment horizons of retailers and producers.

A brief motivation and description of the information approach was given in Section 1.2. This approach is the subject of this thesis and features a new extended relationship between spot and forward prices in which forward looking information known to the market is taken into consideration (c.f. Proposition 4.2.2). As mentioned above, it was motivated by Benth and Meyer-Brandis (2009). In the following sections this will be covered in more detail.

Before, though, we will briefly advise the reader of three papers which are related to our approach thematically if not so much methodologically. The following two papers apply the ideas presented in Benth and Meyer-Brandis (2009) empirically (and rather intuitively) for different markets and underlyings: On the one hand, Ritter et al. (2011) conduct an empirical study on prices of weather derivatives. They compare a standard time series model for temperature with a model that also takes weather forecasts into consideration. These forecasts are incorporated as additional information via an enlarged filtration. They find that forecasts have a significant impact for the last two months before delivery and also that their extended model yields more accurate prices for weather derivatives. On the other hand, Füss et al. (2012) present a fundamental model for electricity spot prices that takes the marginal fuel, demand and capacity into consideration. For the pricing of forward contracts the authors then include demand and capacity forecasts in a very similar way as in Ritter et al. (2011). They conduct an empirical investigation for the British market (gas being the marginal fuel, forecasts provided by the National Grid) and conclude that forward looking information can reduce pricing errors, in particular when reserve margins are tight. Both papers are mainly empirical investigations and do not discuss or extend the mathematical framework.

Last but not least, Cartea et al. (2009) present a related idea. The authors suggest a spot model which takes specific forward looking information into account. This is achieved by introducing a sort of regime-switching component. Its state is determined deterministically by a function of demand and capacity forecasts featuring large spikes only when forecasts predict critical market situations. The authors' emphasis lies on simulating the spot rather than relating the fundamental market objects or pricing forward contracts.

4.2. Spot-Forward Relationships and Premia

What is the relationship between the two fundamental prices on commodity markets (or, more generally, on financial markets), the spot and the forward? There is a huge amount of literature on this issue and the discussion dates back to the 30s and 40s of the last century. Before we start the discussion, we remark that in this chapter we will only consider contracts possessing a delivery time point rather than a delivery period, the reason being readability and conformance with Benth and Meyer-Brandis (2009) (which will be summarised in Section 4.3).

In a perfect world, the forward price would have to be the current spot price adjusted by interest, i.e.

$$F(t,T) = e^{r(T-t)}S_t \tag{4.1}$$

Here, the notation is as in Notation 1.2.1 and r is the interest rate. Still, this relationship is not observable in reality, in particular not for commodities as underlyings.

The *theory of storage* as introduced by Kaldor (1939, pages 5 et seq.) and Working (1949, page 1260) tries to explain the difference between expected spot prices and forward prices using notions like convenience yield, storage costs, warehousing costs etc. In the light of this theory Equation 4.1 can be adjusted to:

$$F(t,T) = e^{(r-y)(T-t)}S_t \tag{4.2}$$

where y denotes costs and yields.

A different approach is the one followed by John Maynard Keynes and John Hicks who argue that market expectations and market position of hedgers and speculators determine which side pays a premium and whether the forward price goes into backwardation or contango (cf. Keynes (1931, Chapter 29.V) and Hicks (1939, page 138)). Keynes coins the term *normal backwardation*.

Following this idea, our starting point will first be the *rational expectation hypothesis*, introduced by Muth (1961). It states that the forward price should equal the expectation of the spot price plus some error term. In probabilistic terms this is

$$F(t,T) = \mathbb{E}^{\mathbb{P}}[S_T \mid \mathcal{F}_t] + \epsilon_t \tag{4.3}$$

where $\mathbb{E}^{\mathbb{P}}[\cdot \mid \mathcal{F}_t]$ denotes the conditional expectation under the real-world measure and the historical filtration and ϵ_t is the error term. Although this error term is, in the purest form of the rational expectation theory (cf. Muth (1961, page 318)), assumed to be unbiased noise there is a multitude of studies that find a premium added to (or subtracted from) the expectation. A literature overview was provided in Section 4.1.

Now, Equation 4.3 allows to formally define the important concept of the *risk premium*:

Definition 4.2.1. Risk Premium. The risk premium $R(t,T)$ is defined to be the difference between the forward price and the expected spot price. In mathematical terms this translates to:

$$R(t,T) = F(t,T) - \mathbb{E}^{\mathbb{P}}[S_T \mid \mathcal{F}_t]$$

where again $\mathbb{P}$ is the real-world measure and $\mathcal{F}_t$ is the historical filtration.

In practice, many papers calculate the expectation of the spot and take the observed forward price to identify the risk premium. In the following, we will always use:

Notation 4.2.1. Observed forward price. We denote the forward price observed on the market by $\hat{F}(t,T)$. We will denote observed versions of objects in this way from now on.

The empirical version of the risk premium would thus be:

$$\hat{R}(t,T) = \hat{F}(t,T) - \mathbb{E}^{\mathbb{P}}[S_T \mid \mathcal{F}_t] \tag{4.4}$$

As mentioned in the previous Section 4.1, the risk premium is the subject of intense research and closely related to the subject of both Benth and Meyer-Brandis (2009) and this thesis.

Going back to Equation 4.3 we formally state (as in Equation 1.1):

Proposition 4.2.1. Classical spot-forward relationship. *The classical spot-forward relationship says that*

$$F(t,T) = \mathbb{E}^{\mathbb{Q}}[S_T \mid \mathcal{F}_t]$$

where $\mathbb{Q}$ is a risk-neutral measure and $\mathcal{F}_t$ is the historical filtration.

Note that this probabilistic version of the spot-forward relationship brings together both Keynes' ideas as well as Equation 4.2.

With Proposition 4.2.1 we can write the following (theoretical) version of the risk premium:

$$R(t,T) = \mathbb{E}^{\mathbb{Q}}[S_T \mid \mathcal{F}_t] - \mathbb{E}^{\mathbb{P}}[S_T \mid \mathcal{F}_t] \tag{4.5}$$

In Financial Mathematics, the general approach to identify or quantify the risk premium is a measure change. This measure change is associated with a drift which is called *market price of risk* in the literature (for example in Bingham and Kiesel (2004, page 245)).

Since we want to study the non-storability property and the impact of information asymmetry on electricity markets (cf. Section 1.2), we introduce further filtrations finer than the historical filtration. For modelling purposes we need, as in Benth and Meyer-Brandis (2009) (whose notation we will adapt), a filtration which contains perfect information (for example on future spot prices) and a slightly coarser filtration which contains some un-specified additional information. To be precise:

Definition 4.2.2. Market and specific filtration. Let $\mathcal{H}_t$ be a filtration which includes the historical filtration as well as some perfect information about the future. Furthermore, let $\mathcal{G}_t$ be a filtration that includes the historical filtration and some (non-perfect, but related) information about the future. We will call this filtration the *market filtration* and we will assume that it represents the relevant information available to market traders.

Clearly, this definition yields the relationship $\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{H}_t$. Also, we remark that this is basically Definition 2.2.1 interpreted economically. We will see concrete examples of the different filtrations in the following sections but the main idea of Benth and Meyer-Brandis (2009) is to add to the market filtration $\mathcal{G}_t$ some knowledge about the future spot price. For example, remembering Benth and Meyer-Brandis (2009, page 126):

Example 4.2.1. Future spot price market filtration. Let $\mathcal{H}_t$ be the filtration with perfect knowledge of the value of the spot in some future time point $T_\Upsilon > t$, i.e.

$$\mathcal{H}_t = \mathcal{F}_t \vee \sigma(S_{T_\Upsilon})$$

As defined above, we assume $\mathcal{G}_t \subseteq \mathcal{H}_t$. For example,

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\mathbb{1}_{S_{T_\Upsilon} > K})$$

so that the market knows that the spot price will be larger than some threshold level K.

Obviously, future information becomes part of the history of the spot once it has passed, i.e. $\mathcal{F}_t = \mathcal{G}_t = \mathcal{H}_t \; \forall t > T_\Upsilon$. With these ideas and definitions at hand we can now propose the following new spot-forward relationship for electricity markets:

Proposition 4.2.2. New spot-forward relationship. *The relationship between spot and forward is given by*

$$F(t,T) = \mathbb{E}^{\mathbb{Q}}[S_T \mid \mathcal{G}_t]$$

where $\mathbb{Q}$ is a risk-neutral measure and $\mathcal{G}_t$ is the market filtration.

As in Benth and Meyer-Brandis (2009) and Equation 1.2, we can now formally define the *information premium*. To be unambiguous, we will always add measure and filtration to the notation of forwards and premia in the following.

Definition 4.2.3. Information Premium. Let $\mathcal{G}_t$ be the market filtration with extra information at T_Υ. Then the information premium is defined as

$$I^{\mathbb{Q}}_{\mathcal{G}}(t,T;T_\Upsilon) = F^{\mathbb{Q}}_{\mathcal{G}}(t,T) - F^{\mathbb{Q}}_{\mathcal{F}}(t,T)$$

i.e. the difference between the forward prices as calculated under the market and the historical filtration.

We are going to assume that all market participants work with the filtration $\mathcal{G}$. This implies that instead of assuming observed forward prices to satisfy Proposition 4.2.1 we rather believe that they are calculated by market participants via

$$\hat{F}(t,T) = \mathbb{E}^{\mathbb{Q}}[S_T|\mathcal{G}_t] \tag{4.6}$$

The information approach does neither make the risk premium literature obsolete nor contradict it. Rather, it adds another aspect to the fundamental discussion. To see this, we relate the two premia in the following lemma (cf. Benth and Meyer-Brandis (2009, page 116)):

Lemma 4.2.1. Relation of the premia. *We have the following relationship between the information premium and the risk premium:*

$$R^{\mathbb{Q}}_{\mathcal{F}}(t,T) = R^{\mathbb{Q}}_{\mathcal{G}}(t,T) - I^{\mathbb{Q}}_{\mathcal{G}}(t,T) + I^{\mathbb{P}}_{\mathcal{G}}(t,T)$$

Proof. This is a direct consequence of the definitions. □

This means that the traditional risk premium is the sum of both, the risk premium under the market filtration as well as the influence from additional future information observable by market participants. The latter part of the risk premium is quantified by the difference between the information premia under the relevant measures.

Benth and Meyer-Brandis (2009, page 117) also propose another characterisation of the information premium. They see it as the residual of projecting the $\mathcal{G}$-forward price onto the space spanned by the historical filtration. This can be easily justified by the following lemma:

Lemma 4.2.2. Orthogonality property. *The information premium is the residual of projecting the forward price under $\mathcal{G}_t$ onto the space $L^2(\mathcal{F}_t, \mathbb{Q})$. In other words*

$$\mathbb{E}^{\mathbb{Q}}[I_{\mathcal{G}}(t,T)|\mathcal{F}_t] = 0$$

Proof. Simple application of the tower property. □

It is this rather trivial result that will pose the biggest problem when trying to identify the information premium empirically (cf. Chapter 8). Clearly, Lemma 4.2.2 holds for all equivalent measures which in turn means that the information premium cannot be attained by the usual method of changing measure.

4.3. The Model of Benth/Meyer-Brandis

In this section we will summarise parts of the calculations of Benth and Meyer-Brandis (2009), which is the motivating paper of this thesis. We will rely heavily on the concepts and ideas from the preceding section as well as the properties of the two-factor arithmetic spot model introduced in Chapter 3. Moreover, we will leave out most proofs here as almost all results will be proved as special cases of more general results in Chapter 5.

The starting point of the paper by Benth and Meyer-Brandis is the same as in Section 1.2 and in particular, the authors give the introduction of CO_2 certificates as a real-world example. The paper then describes how to calculate the information premium.

Given additional information about the future spot price and the model given by Equation 3.2, i.e. $S_t = \Lambda_t + X_t + Y_t$, we have two possibilities mathematically. On the one hand, there is future information about the Lévy component Y_t. This would correspond to the knowledge of short outages or similar shock-like events. On the other hand, one could have information about the base component X_t of the spot. This represents, for example, non-standard weather forecasts or longer plant maintenance. The theory of the *enlargement of filtrations* presented in Section 2.2 provides tools to deal with both additional Brownian and Lévy information. Still, enlarging by the future value of the two components corresponds to enlarging the filtration by the value of a functional of the driving processes and this is only possible in closed form for the Brownian case (or, more generally, when the density of the functional is known). In the Lévy case we will need to be given the value of L_{T_Υ} rather than Y_{T_Υ}. This is more restrictive in the sense that it necessitates possessing knowledge about the driving process of a spot component rather than about the component as a whole.

Before we commence, we will introduce the notion of the *information yield* as defined by Benth and Meyer-Brandis (2009, page 118):

Definition 4.3.1. Information yield. The information yield of a Lévy process L_t is the drift process $\mu_t^{\mathcal{G}}$ that makes

$$\xi_t = L_t - \int_0^t \mu_s^{\mathcal{G}} ds$$

a $\mathcal{G}_t$-martingale. The drift $\mu_t^{\mathcal{G}}$ is $\mathcal{G}_t$-measurable.

This object plays a similar role under enlargement of filtration as the *market price of risk* does when changing measure. The name was chosen by the authors as to remind of the *convenience yield*. Clearly, $\mu_t^{\mathcal{G}}$ also reminds us of Section 2.2 and the decompositions under an enlarged filtration.

Finally, we remark that Benth and Meyer-Brandis (2009) only consider forward contracts without delivery period and under the real-world measure $\mathbb{P}$.

4.3.1. Future Lévy Information

We consider the filtration

$$\mathcal{G}_t \subseteq \mathcal{H}_t = \mathcal{F}_t \vee \sigma(L_{T_\Upsilon}) \tag{4.7}$$

For this scenario the information yield is readily given by Theorem 2.2.1 and Corollary 2.2.1:

$$L(t) - \int_0^t \frac{\mathbb{E}[L_{T_\Upsilon} - L_s \mid \mathcal{G}_t]}{T_\Upsilon - s} ds \tag{4.8}$$

This, together with a number of auxiliary results, allows to calculate the information premium (cf. Benth and Meyer-Brandis (2009, Proposition 3.2)).

Proposition 4.3.1. Information premium (delivery point, spike component). *With the market filtration as defined above and $t \leq T \leq T_\Upsilon$ the information premium for a forward in t with maturity T is given by:*

$$I_{\mathcal{G}}^{\mathbb{P}}(t,T;T_\Upsilon) = \frac{1}{\beta}\left(\frac{\mathbb{E}[L_{T_\Upsilon} - L_t \mid \mathcal{G}_t]}{T_\Upsilon - t} + \psi'_{L_1}(0)\right)\left(1 - e^{-\beta(T-t)}\right)$$

Proof. The proof is provided in Lemma 5.3.2 as the limit behaviour of Proposition 5.3.1. □

We can easily see that parts of this expression stem from Proposition 3.2.1, i.e. the forward price under $\mathcal{F}_t$. We consider an arithmetic model and both filtrations coincide for the base component as well as (trivially) for the deterministic terms, hence they cancel.

It has been assumed in the proposition that $t \leq T \leq T_\Upsilon$, i.e. that the additional information is located after the maturity of the forward. The following corollary will provide the information premium for the case that the maturity is after the additional information.

Corollary 4.3.1. Information premium (delivery point, spike component). *For $t \leq T_\Upsilon \leq T$ the information premium is given by:*

$$I_{\mathcal{G}}^{\mathbb{P}}(t,T;T_\Upsilon) = I_{\mathcal{G}}^{\mathbb{P}}(t,T_\Upsilon;T_\Upsilon)e^{-\beta(T-T_\Upsilon)}$$

Proof. Provided by Lemma 5.3.3. □

Clearly, when conducting calculations for forward contracts with delivery period there will be more than two and more complicated cases (future information before, during and after the delivery period).

Let us provide some interpretation for these formulae. First of all, we can take limits

$$\lim_{T\to\infty} I_{\mathcal{G}}^{\mathbb{P}}(t,T;T_\Upsilon) = \lim_{T\to\infty} e^{-\beta(T-T_\Upsilon)} I_{\mathcal{G}}^{\mathbb{P}}(t,T_\Upsilon;T_\Upsilon) = 0 \tag{4.9}$$

i.e. the information premium tends to zero whenever the end of the contract is very far in the future. This is in line with economic intuition. Also, the sign of the premium stays the same for all scenarios in which $T > T_\Upsilon$.

Considering, for example, threshold-type additional future information (cf. Benth and Meyer-Brandis (2009, page 126)), i.e.

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma\left(\mathbb{1}_{\{L_{T_\Upsilon} \leq K\}}\right) \tag{4.10}$$

this could correspond to events like sudden power plant outages or an updated short-term weather report where we only know that the future spot will be larger or smaller than price K.

We can rewrite the expression of Proposition 4.3.1 according to

$$\begin{aligned}I_{\mathcal{G}}^{\mathbb{P}}(t,T;T_\Upsilon) &= \frac{1-e^{-\beta(T-t)}}{\beta(T_\Upsilon-t)}\left(\mathbb{E}[L_{T_\Upsilon}-L_t \mid \mathcal{G}_t] - \mathbb{E}[L_{T_\Upsilon}-L_t]\right)\\ &= \frac{1-e^{-\beta(T-t)}}{\beta(T_\Upsilon-t)}\Big(\left(\mathbb{E}[L_{T_\Upsilon}-L_t \mid L_{T_\Upsilon}\le K] - \mathbb{E}[L_{T_\Upsilon}-L_t]\right)\mathbb{1}_{\{L_{T_\Upsilon}\le K\}}\\ &\qquad + \left(\mathbb{E}[L_{T_\Upsilon}-L_t \mid L_{T_\Upsilon}> K] - \mathbb{E}[L_{T_\Upsilon}-L_t]\right)\mathbb{1}_{\{L_{T_\Upsilon}> K\}}\Big)\end{aligned}$$

Benth and Meyer-Brandis (2009, page 127) rightly point out that with this expression we can identify the sign of the information premium. If for example $L_{T_\Upsilon} \le K$, then the unconditional expectation is greater than the conditional one and thus, the premium becomes negative. Ignoring the additional information would yield a too large price and possible losses.

4.3.2. Future Brownian Information

Now, we will consider the filtration enlarged by the value of the base component as a whole

$$\mathcal{G}_t \subseteq \mathcal{H}_t = \mathcal{F}_t \vee \sigma(X_{T_\Upsilon}) = \mathcal{F}_t \vee \sigma\left(\int_0^{T_\Upsilon} e^{\alpha s} dW_s\right) \tag{4.11}$$

The information yield (cf. Definition 4.3.1) can be calculated using Theorem 2.2.4. Benth and Meyer-Brandis (2009, Equation 33) find the decomposition under $\mathcal{G}_t$ using a slightly different result, which we provide in the appendix (cf. Theorem A.2):

$$\xi_t = W_t - \int_0^t \frac{2\alpha e^{\alpha s}}{e^{2\alpha T_\Upsilon} - e^{2\alpha s}}\, \mathbb{E}\left[\int_s^{T_\Upsilon} e^{\alpha u} dW_u \mid \mathcal{G}_s\right] ds \tag{4.12}$$

It is possible to write the information yield in terms of the base component X_t. Making good use of auxiliary results Benth and Meyer-Brandis (2009, Equation 36) state:

Proposition 4.3.2. Information premium (delivery point, base component). *For $t \le T \le T_\Upsilon$ the information premium is given by:*

$$I_{\mathcal{G}}^{\mathbb{P}}(t,T;T_\Upsilon) = e^{\alpha(T_\Upsilon - T)} \frac{e^{2\alpha T} - e^{2\alpha t}}{e^{2\alpha T_\Upsilon} - e^{2\alpha t}} \left(\mathbb{E}\left[X_{T_\Upsilon} \mid \mathcal{G}_t\right] - e^{-\alpha(T_\Upsilon - t)} X_t\right)$$

Proof. Provided by Lemma 5.3.5. □

Similar to the Lévy case a corollary provides the information premium for the case $t \le T \le T_\Upsilon$ as a transformed expression of the result of Proposition 4.3.2.

Corollary 4.3.2. Information premium (delivery point, base component). *For $t \le T_\Upsilon \le T$ the information premium takes the form*

$$I_{\mathcal{G}}^{\mathbb{P}}(t,T;T_\Upsilon) = e^{-\alpha(T-T_\Upsilon)} I_{\mathcal{G}}^{\mathbb{P}}(t,T_\Upsilon;T_\Upsilon)$$

Proof. Again, provided by Lemma 5.3.5. □

4.3.3. Further Calculations

Benth and Meyer-Brandis (2009) briefly discuss two more ideas: firstly, they provide a first result on how to calculate the information premium when given spot information at several future dates $T_{\Upsilon_1}, T_{\Upsilon_2}, \ldots$ rather than one isolated T_{Υ} . They find that if one splits up expressions carefully and assumes that $\mathcal{G} = \mathcal{H}$ closed-form results are possible. We will discuss this issue in Section 5.4.

Secondly, they propose a simple process for the temperature and let the additional information be given as weather forecasts. They relate the spot price process with temperature by introducing correlation between the driving processes. In this case, too, they find a closed-form solution for the information premium. We will provide details in Section 5.5.

Chapter 5.

The Information Premium with Delivery Period

5.1. Literature Overview and Summary

In this chapter we will provide formulae for the information premium as defined previously (cf. Definition 4.2.3) and for different scenarios. These scenarios include information about different components of the spot price model or even separate but correlated processes. As an example of the latter we will introduce, in Section 5.5, a stochastic model for the temperature which we will correlate with the base component of the spot process. A number of relevant papers will be mentioned and discussed. Furthermore, we will discuss the information premium for different arrangements of the time axis, in particular for the case that more than one piece of future information is known to the market. We will prove the formulae of Benth and Meyer-Brandis (2009) (summarised in Section 4.3) as special cases of more general results. Stylised examples will illustrate the different formulae graphically and we will interpret economically. First, though, we will apply the mathematical tools from Section 2.2 and the literature cited therein to identify the information yield.

5.2. Calculating the Information Yield

We defined the notion of the *information yield* in Definition 4.3.1. Furthermore, we mentioned the equivalence between the information yield and the drift process $\mu_t^{\mathcal{G}}$ related to filtration enlargement. This drift was discussed in Section 2.2. In this section we will calculate the information yield for different types of future information: knowledge about the future value of the driving process, knowledge about the future value of the spot component and the special case of the threshold information (as discussed in Section 4.3). The calculations will be done applying different methods: firstly, we will use Itō's Theorem (cf. Theorem 2.2.1) for easy cases. Secondly, in case we know distributions, we will apply Jacod's criterion (this is Assumption 2.2.2 and Theorem 2.2.4). We remark that a special case of Jacod's criterion is the theorem proposed by Jeulin and Chalevat-Maurel (cf. Theorem A.2). Thirdly, we discussed Imkeller's method in Section 2.2.3 and we will use it as an easier way for more complicated cases. Again, we will denote by T_Υ the future date with additional information available.

5.2.1. Additional Information about the Lévy Component

We have discussed before that closed-form solutions for enlargement of filtration by functionals of a process are only possible in case we know the distribution of the functional. Still, there is no closed-form expression for the distribution of the

Lévy process chosen in Definition 3.3.2 and thus not for the jump component Y_t (cf. Papapantoleon (2008, page 34)). Hence, we can only find the information yield for the case that some information about the future value of the Lévy process L is known and we have already seen in Equation 4.8 that the drift for $s > t$ is given by:

$$\mu_s^{\mathcal{G}} = \frac{\mathbb{E}[L_{T_\Upsilon} - L_s \mid \mathcal{G}_t]}{T_\Upsilon - s} \tag{5.1}$$

A brute force method to apply Jacod's criterion would be to use the characteristic function of component Y_t (which we can calculate, we refer to Benth et al. (2008b, Proposition 3.2)) and apply one of the inversion formulae. Sadly, no closed-form solution can be found either and one would have to resolve to numerics. We propose this as a topic of future research.

5.2.2. Additional Information about the Brownian Motion

In this section we will assume that the market filtration satisfies $\mathcal{G}_t \subseteq \mathcal{H}_t = \mathcal{F}_t \vee \sigma(W_{T_\Upsilon})$. As before, we can trivially apply Itō's theorem (cf. Theorem 2.2.1) for $\mathcal{G}_t = \mathcal{H}_t$. Non-precise information is a special case and the result is given by Corollary 2.2.1. Let $s > t$, then the information yield is

$$\mu_s^{\mathcal{G}} = \frac{\mathbb{E}[W_{T_\Upsilon} - W_s \mid \mathcal{G}_t]}{T_\Upsilon - s} \tag{5.2}$$

We will now briefly describe how to arrive at this result making use of Jacod's criterion and Imkeller's method. This will make more complicated cases easier to understand.

5.2.2.1. Brownian motion information: Jacod's criterion

In this case, the random variable G by which we enlarge the filtration $\mathcal{F}_t$ is W_{T_Υ}. This is the standard example of an application of Jacod's criterion and the calculation is conducted, amongst others, in Jeanblanc (2010, Page 29) and Mansuy and Yor (2006). Clearly, G is normally distributed with law

$$P^G(dl) = \frac{1}{\sqrt{2\pi T_\Upsilon}} \exp\left(-\frac{l^2}{2T_\Upsilon}\right) dl$$

The conditional law of G given the historical filtration $\mathcal{F}_t$ is

$$\begin{aligned} P_t^G(\omega, dl) &= \mathbb{P}(W_{T_\Upsilon} \in dl | \mathcal{F}_t) = \mathbb{P}(W_{T_\Upsilon} - W_t \in dl - W_t) \\ &= \frac{1}{\sqrt{2\pi(T_\Upsilon - t)}} \exp\left(-\frac{(l - W_t)^2}{2(T_\Upsilon - t)}\right) dl \end{aligned}$$

Now, Jacod's condition (A') (i.e. Assumption 2.2.3) says that $P_t^G(\omega, dl)$ needs to be absolutely continuous with respect to $P^G(dl)$ which corresponds to the existence of the Radon-Nikodým density process

$$p_t(\omega, l) = \frac{dP_t^G(\omega, \cdot)}{dP^G(\cdot)}(l) = \sqrt{\frac{T_\Upsilon}{(T_\Upsilon - t)}} \exp\left(-\frac{1}{2}\frac{(l - W_t)^2}{T_\Upsilon - t} + \frac{1}{2}\frac{l^2}{T_\Upsilon}\right)$$

We will now check the martingale property of p_t and find its canonical decomposition. Obviously, we have $p_0 = 1$. Then we apply Itō's lemma with function $f(t, x)$ given by

$$\begin{aligned} f(t,x) &= \sqrt{\frac{T_\Upsilon}{(T_\Upsilon - t)}} \exp\left(-\frac{1}{2}\frac{(l-x)^2}{T_\Upsilon - t} + \frac{1}{2}\frac{l^2}{T_\Upsilon}\right) \\ f_x(t,x) &= f(t,x)\frac{(l-x)}{T_\Upsilon - t} \\ f_{xx}(t,x) &= f(t,x)\frac{(l-x)^2}{(T_\Upsilon - t)^2} - f(t,x)\frac{1}{T_\Upsilon - t} \\ f_t(t,x) &= \frac{1}{2}\frac{1}{T_\Upsilon - t}f(t,x) - \frac{1}{2}f(t,x)\frac{(l-x)^2}{(T_\Upsilon - t)^2} \end{aligned}$$

where $x = W_t$, drift $m = 0$ and variance $s^2 = 1$. Then, the drift term of $df(t, x)$ cancels providing

$$dp_t(\omega, l) = p_t \frac{(l - W_t)}{T_\Upsilon - t} dW_t$$

which is a martingale. Integrating yields

$$p_t = p_0 + \int_0^t p_s \frac{W_{T_\Upsilon} - W_s}{T_\Upsilon - s} dW_s$$

According to Theorem 2.2.4 we now compute the quadratic variation of p_t and W_t:

$$d < p_., W_. >_t = d < \int_0^{\cdot} p_s \frac{W_{T_\Upsilon} - W_s}{T_\Upsilon - s} dW_s, \int_0^{\cdot} dW_s >_t = p_t \frac{W_{T_\Upsilon} - W_t}{T_\Upsilon - t}$$

Finally, we construct the information yield under the precise filtration $\mathcal{H}_t$

$$\mu_t^{\mathcal{H}} = \frac{d < p_., W_. >_t}{p_t} = \frac{W_{T_\Upsilon} - W_t}{T_\Upsilon - t}$$

Not surprisingly, this result agrees with the previous one (again, an application of Corollary 2.2.1 makes it possible to change from precise $\mathcal{H}_t$ to market $\mathcal{G}_t$).

5.2.2.2. Brownian motion information: Imkeller's method

We have seen in the previous section that finding the canonical decomposition of process p_t is tedious. Theorem 2.2.8 provided Imkeller's result connecting enlargement of filtration and Malliavin calculus.

According to this theorem, the information drift $\mu_t^{\mathcal{H}}$ can be calculated via

$$\mu_t^{\mathcal{H}} = \frac{\mathcal{D}_t P_t^G(\cdot, dl)}{P_t^G(\cdot, dl)}(l)$$

where P^G and P_t^G are again as in Section 5.2.2.1 and $\mathcal{D}_t$ denotes the Malliavin derivative in t. With our current example we find:

$$\begin{aligned}\mathcal{D}_t P_t^G &= \mathcal{D}_t\left(\frac{1}{\sqrt{2\pi(T_\Upsilon - t)}}\exp\left(-\frac{(l-W_t)^2}{2(T_\Upsilon - t)}\right)\right) = P_t^G\mathcal{D}_t\left(-\frac{(l-W_t)^2}{2(T_\Upsilon - t)}\right)\\ &= -P_t^G\frac{l-W_t}{T_\Upsilon - t}\mathcal{D}_t\left(-\int_0^t dW_s\right) = P_t^G\frac{l-W_t}{T_\Upsilon - t}\end{aligned}$$

Dividing by P_t^G and setting $l = W_{T_\Upsilon}$ yields the same $\mu_t^{\mathcal{H}}$ as expected and as before. Note that we have exploited properties of *Wiener polynomials* (in the last step, cf. Definition 2.2.2) and utilised the chain rule of the Malliavin derivative (cf.Theorem 2.2.7).

5.2.3. Additional Information about the Base Component

In this section, we will assume that we are given the value of the base component X_{T_Υ}. The market filtration $\mathcal{G}$ then satisfies $\mathcal{G}_t \subseteq \mathcal{H}_t = \mathcal{F}_t \vee \sigma(X_{T_\Upsilon})$. As usual, we will consider the $\mathcal{H}_t$-case first and then apply Corollary 2.2.1. Now, we enlarge by a functional of the underlying Brownian motion. The following equality holds (given $\mathcal{F}_t$):

$$\sigma(X_{T_\Upsilon}) = \sigma\left(e^{-\alpha(T_\Upsilon - t)}X_t + \sigma\int_t^{T_\Upsilon} e^{-\alpha(T_\Upsilon - s)}dW_s\right) = \sigma\left(\int_t^{T_\Upsilon} e^{-\alpha(T_\Upsilon - s)}dW_s\right)$$

Thus, we are enlarging by random variable

$$G = \int_0^{T_\Upsilon} e^{-\alpha(T_\Upsilon - s)}dW_s \tag{5.3}$$

Again, as an Itō integral, this is normally distributed with zero mean and variance

$$s_{T_\Upsilon}^2 = \frac{1}{2\alpha}\left(1 - e^{-2\alpha T_\Upsilon}\right)$$

Thus,

$$P^G(dl) = \frac{1}{\sqrt{2\pi s_{T_\Upsilon}^2}}\exp\left(-\frac{1}{2}\frac{l^2}{s_{T_\Upsilon}^2}\right)dl \tag{5.4}$$

For the conditional distribution we define auxiliary variables

$$m_t = \int_0^t e^{-\alpha(T_\Upsilon - s)}dW_s \tag{5.5}$$

$$s_t^2 = Var\left(\int_0^t e^{-\alpha(T_\Upsilon - s)}dW_s\right) = \frac{1}{2\alpha}\left(e^{-2\alpha(T_\Upsilon - t)} - e^{-2\alpha T_\Upsilon}\right) \tag{5.6}$$

and calculate

$$\begin{aligned} P_t^G(dl) &= \mathbb{P}\left(\int_0^{T_\Upsilon} e^{-\alpha(T_\Upsilon - s)} dW_s - m_t \in dl - m_t\right) \\ &= \frac{1}{\sqrt{2\pi(s_{T_\Upsilon}^2 - s_t^2)}} \exp\left(-\frac{1}{2}\frac{(l-m_t)^2}{s_{T_\Upsilon}^2 - s_t^2}\right) dl \end{aligned} \tag{5.7}$$

Now, we will again apply both Jacod's criterion and Imkeller's method.

5.2.3.1. Base component information: Jacod's criterion

As before, we commence by finding the process p_t, dividing Equation 5.7 by Equation 5.4:

$$p_t(\omega, l) = \frac{dP_t^G(\omega, \cdot)}{dP^G(\cdot)}(l) = \sqrt{\frac{s_{T_\Upsilon}^2}{(s_{T_\Upsilon}^2 - s_t^2)}} \exp\left(-\frac{1}{2}\frac{(l-m_t)^2}{s_{T_\Upsilon}^2 - s_t^2} + \frac{1}{2}\frac{l^2}{s_{T_\Upsilon}^2}\right)$$

Now, we need the canonical decomposition of this process. We use Itō's lemma again and this is the tedious part of the method. We choose the function

$$f(t,x) = \sqrt{\frac{s_{T_\Upsilon}^2}{(s_{T_\Upsilon}^2 - s_t^2)}} \exp\left(-\frac{1}{2}\frac{(l-x)^2}{s_{T_\Upsilon}^2 - s_t^2} + \frac{1}{2}\frac{l^2}{s_{T_\Upsilon}^2}\right)$$

and a rather long calculation yields derivatives

$$\begin{aligned} f_t(t,x) &= \frac{1}{2}\frac{f(t,x)}{s_{T_\Upsilon}^2 - s_t^2} e^{-2\alpha(T_\Upsilon - t)} \left(1 - \frac{(l-x)^2}{s_{T_\Upsilon}^2 - s_t^2}\right) \\ f_x(t,x) &= \frac{p_t}{(s_{T_\Upsilon}^2 - s_t^2)}(l - m_t) \\ f_{xx}(t,x) &= \frac{p_t}{(s_{T_\Upsilon}^2 - s_t^2)}\left(\frac{(l-m_t)^2}{s_{T_\Upsilon}^2 - s_t^2} - 1\right) \end{aligned}$$

The dynamics of p_t are then given by

$$dp_t = p_t \frac{l - m_t}{s_{T_\Upsilon}^2 - s_t^2} e^{-\alpha(T_\Upsilon - t)} dW_t \tag{5.8}$$

which is a martingale. Thus, by Jacod's criterion (Assumption 2.2.3 and Theorem 2.2.4) we get the information yield

$$\mu_t^{\mathcal{H}} = \frac{G - m_t}{s_{T_\Upsilon}^2 - s_t^2} e^{-\alpha(T_\Upsilon - t)} \tag{5.9}$$

We can now state the decomposition of W_t:

Proposition 5.2.1. Decomposition of the Brownian motion. *Under filtration $\mathcal{H}_t = \mathcal{F}_t \vee \sigma(X_{T_\Upsilon})$ the decomposition of the Brownian motion W_t is given by*

$$W_t = \xi_t + \int_0^t \left(\int_s^{T_\Upsilon} e^{\alpha u} dW_u \right) \frac{2\alpha e^{\alpha s}}{e^{2\alpha T_\Upsilon} - e^{2\alpha s}} ds$$

where ξ_t is an $\mathcal{H}_t$-Brownian motion.

Proof. Straightforward calculation plugging in Equation 5.3 and integrating the expression from Equation 5.9:

$$\begin{aligned} W_t &= \xi_t + \int_0^t \frac{G - m_s}{s^2_{T_\Upsilon} - s^2_s} e^{-\alpha(T_\Upsilon - s)} ds \\ &= \xi_t + \int_0^t \frac{\int_s^{T_\Upsilon} e^{-\alpha(T_\Upsilon - u)} dW_u}{\frac{1}{2\alpha}\left(1 - e^{-2\alpha(T_\Upsilon - s)}\right)} e^{-\alpha(T_\Upsilon - s)} ds \\ &= \xi_t + \int_0^t \left(\int_s^{T_\Upsilon} e^{\alpha u} dW_u \right) \frac{2\alpha e^{\alpha s}}{e^{2\alpha T_\Upsilon} - e^{2\alpha s}} ds \end{aligned}$$

as required. □

This is the same expression as the one found in Benth and Meyer-Brandis (2009) (cf. Equation 4.12, the authors use a less general result proposed by Jeulin, Theorem A.2, to identify the information drift). Furthermore, for practical purposes it is more suitable to know the decomposition in terms of X rather than W:

Corollary 5.2.1. Decomposition in terms of the base component. *The decomposition of W_t under $\mathcal{H}_t$ in terms of the base component X is given by*

$$W_t = \xi_t + \int_0^t \frac{1}{\sigma} e^{\alpha T_\Upsilon} \left(X_{T_\Upsilon} - e^{-\alpha(T_\Upsilon - s)} X_s \right) \frac{2\alpha e^{\alpha s}}{e^{2\alpha T_\Upsilon} - e^{2\alpha s}} ds$$

Proof. We can multiply by one and add zero to the expression from Proposition 5.2.1:

$$\begin{aligned} \int_s^{T_\Upsilon} e^{\alpha u} dW_u &= e^{\alpha T_\Upsilon} \frac{\sigma}{\sigma} \int_s^{T_\Upsilon} e^{-\alpha(T_\Upsilon - u)} dW_u \\ &= e^{\alpha T_\Upsilon} \frac{1}{\sigma} \left(e^{-\alpha(T_\Upsilon - u)} X_s - e^{-\alpha(T_\Upsilon - u)} X_s + \sigma \int_s^{T_\Upsilon} e^{-\alpha(T_\Upsilon - u)} dW_u \right) \end{aligned}$$

which is the required result. □

Another application of Corollary 2.2.1 gives the decomposition under the market filtration $\mathcal{G}_t$:

Corollary 5.2.2. Decomposition under the market filtration. *The decomposition of the Brownian motion under filtration $\mathcal{G}_t \subseteq \mathcal{F}_t \vee \sigma(X_{T_\Upsilon})$ is given by*

$$W_t = \xi_t + \int_0^t \mathbb{E}\left[\int_s^{T_\Upsilon} e^{\alpha u} dW_u \mid \mathcal{G}_s \right] \frac{2\alpha e^{\alpha s}}{e^{2\alpha T_\Upsilon} - e^{2\alpha s}} ds$$

or, in terms of X_{T_Υ}

$$W_t = \xi_t + \int_0^t \frac{1}{\sigma} e^{\alpha T_\Upsilon} \left(\mathbb{E}[X_{T_\Upsilon} \mid \mathcal{G}_s] - e^{-\alpha(T_\Upsilon - s)} X_s \right) \frac{2\alpha e^{\alpha s}}{e^{2\alpha T_\Upsilon} - e^{2\alpha s}} ds$$

Here, ξ_t is a $\mathcal{G}_t$-Brownian motion.

5.2.3.2. Base component information: Imkeller's method

Applying Imkeller's method in this case helps simplifying calculations even more than in Section 5.2.2. We need to find the Malliavin derivative of Equation 5.7:

$$\mathcal{D}_t P_t^G = P_t^G \mathcal{D}_t \left(-\frac{1}{2} \frac{(l-m_t)^2}{s_{T_\Upsilon}^2 - s_t^2} \right) = P_t^G \frac{l-m_t}{s_{T_\Upsilon}^2 - s_t^2} \mathcal{D}_t(m_t) = P_t^G \frac{l-m_t}{s_{T_\Upsilon}^2 - s_t^2} e^{-\alpha(T_\Upsilon - t)}$$

where, again, we have used the properties of Wiener polynomials and the Malliavin chain rule. Dividing this expression by P_t^G we arrive, not surprisingly, at the same information yield as Equation 5.9.

5.2.4. Base Component Threshold Information

We can further refine the expression for the information yield if the structure of the market filtration $\mathcal{G}_t$ is known. Remembering Example 4.2.1 we will now examine threshold information, i.e. $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\mathbb{1}_{X_{T_\Upsilon} > K})$. Again, there are two possibilities to calculate the information yield in this case.

5.2.4.1. Threshold information as a special case

Here, our starting point will be the result of Corollary 5.2.2. Introducing short-hand notation

$$a(t) = \frac{2\alpha e^{\alpha t}}{e^{2\alpha T_\Upsilon} - e^{2\alpha t}} \tag{5.10}$$

this is:

$$W_t = \xi_t + \int_0^t \frac{a(s)}{\sigma} e^{\alpha T_\Upsilon} \left(\mathbb{E}[X_{T_\Upsilon} \mid \mathcal{G}_s] - e^{-\alpha(T_\Upsilon - s)} X_s \right) ds$$

Then, we can write

$$\mathbb{E}[X_{T_\Upsilon} \mid \mathcal{G}_s] = \mathbb{E}[X_{T_\Upsilon} | X_{T_\Upsilon} > K, X_s] \mathbb{1}_{X_{T_\Upsilon} > K} + \mathbb{E}[X_{T_\Upsilon} | X_{T_\Upsilon} < K, X_s] \mathbb{1}_{X_{T_\Upsilon} < K}$$

We define two more auxiliary variables:

$$\begin{aligned} \Sigma_s^2 &= \frac{\sigma^2}{2\alpha} \left(1 - e^{-2\alpha(T_\Upsilon - s)} \right) \\ K_s &= \frac{K - e^{-\alpha(T_\Upsilon - s)} X_s}{\Sigma_s} \end{aligned}$$

Furthermore, let Z denote a standard normal random variable. Then:

$$\begin{aligned} \mathbb{E}[X_{T_\Upsilon} \mid X_{T_\Upsilon} > K, X_s] &= X_s e^{-\alpha(T_\Upsilon - s)} + \Sigma_s \mathbb{E}\left[Z \mid Z > K_s \right] \\ &= X_s e^{-\alpha(T_\Upsilon - s)} + \Sigma_s \mathbb{E}\left[Z \mid Z \le -K_s \right] \end{aligned}$$

Under the market filtration Z is a truncated normal random variable from $-\infty$ to K_s. Thus, we need to normalise the density by the distribution (denoted by $\phi(\cdot)$ and $\Phi(\cdot)$ as usual). This allows us to calculate:

$$\mathbb{E}[Z|Z \leq -K_s] = \int_{-\infty}^{-K_s} x\frac{\phi(x)}{\Phi(-K_s)}dx = \Phi^{-1}(-K_s)\int_{-\infty}^{-K_s} x\phi(x)dx = \frac{\phi(K_s)}{\Phi(-K_s)}$$

Conducting the same calculation for the case $X_{T_\Upsilon} < K$ and collecting terms yields the decomposition

$$W_t = \xi_t + \int_0^t e^{\alpha T_\Upsilon}\frac{a(s)}{\sigma}\Sigma_s\left(\frac{\phi(K_s)}{\Phi(-K_s)}\mathbb{1}_{X_{T_\Upsilon}>K} + \frac{-\phi(K_s)}{\Phi(K_s)}\mathbb{1}_{X_{T_\Upsilon}\leq K}\right)ds \tag{5.11}$$

5.2.4.2. Threshold information as a discrete random variable

Another method to obtain the decomposition is to interpret the future information as a discrete random variable $G = \mathbb{1}_{X_{T_\Upsilon}>K}$. We can then, for example, apply Imkeller's method. We need to identify the regular conditional distribution of G under the historical filtration:

$$P_t(G=1) = \mathbb{P}(X_{T_\Upsilon} > K|\mathcal{F}_t) = \mathbb{P}(Z > K_t) = \mathbb{P}(Z < -K_t) = \Phi(-K_t)$$

where Z again denotes a standard normal random variable. We can now calculate the Malliavin derivative of $P_t(G=1)$ and find

$$\begin{aligned}
\mathcal{D}_t P_t(G=1) &= \mathcal{D}_t\Phi(-K_t) = \frac{d\Phi}{dK_t}\mathcal{D}_t(-K_t)\\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{K_t^2}{2}}\mathcal{D}_t\left(-\frac{1}{\Sigma_t}\left(K - e^{-\alpha T_\Upsilon}X_0 - \sigma\int_0^t e^{-\alpha(T_\Upsilon-u)}dW_u\right)\right)\\
&= \phi(K_t)\frac{1}{\Sigma_t}\sigma e^{-\alpha(T_\Upsilon-t)}
\end{aligned}$$

The Malliavin derivative of $P_t(G=0)$ is

$$\mathcal{D}_t P_t(G=0) = \mathcal{D}_t\Phi(K_t) = -\phi(K_t)\frac{1}{\Sigma_t}\sigma e^{-\alpha(T_\Upsilon-t)}$$

Thus, the decomposition of W_t is given by

$$W_t = \xi_t + \int_0^t \sigma e^{-\alpha(T_\Upsilon-s)}\frac{1}{\Sigma_s}\left(\frac{\phi(K_s)}{\Phi(-K_s)}\mathbb{1}_{X_{T_\Upsilon}>K} + \frac{-\phi(K_s)}{\Phi(K_s)}\mathbb{1}_{X_{T_\Upsilon}\leq K}\right)ds$$

and a straightforward calculation shows that this is equal to Equation 5.11 as suspected. Knowing how to calculate the information yield with different techniques we can now proceed to calculate the information premium.

5.3. The Information Premium for Different Setups of the Time Axis

In this section, we will calculate expressions for the information premium (cf. Definition 4.2.3) for forwards with delivery period given the spot model specified in Definition 3.3.1. Remembering the definition of the information premium as the difference between the forwards under the market filtration and the historical filtration we adjust this definition to delivery periods:

Definition 5.3.1. Information premium with delivery period. The information premium in t with delivery period in $[T_1, T_2]$ and future information given in T_Υ is defined as

$$I_{\mathcal{G}}^{\mathbb{Q}}(t, T_1, T_2; T_\Upsilon) = F_{\mathcal{G}}^{\mathbb{Q}}(t, T_1, T_2; T_\Upsilon) - F_{\mathcal{F}}^{\mathbb{Q}}(t, T_1, T_2)$$

where $\mathbb{Q}$ is a (risk-neutral) measure and $\mathcal{G}_t$ is the market filtration.

We remark that we will calculate the information premium directly rather than the forward under the market filtration first and then the difference. The $\mathcal{G}_t$-forward price is then given by adding the information premium to the $\mathcal{F}_t$-forward.

Furthermore, we will provide the information premium for two cases of future information: knowledge about the Lévy process and knowledge about the base component of our spot model. Also, we will have to consider different cases depending on the time of future information relative to the delivery period (similar to Section 4.3, note that the information yield from the last section and corresponding results from Chapter 2 require $T_2 < T_\Upsilon$).

5.3.1. Information Premium with Spike Information

In this section, the market filtration will be given by

$$\mathcal{G}_t \subseteq \mathcal{H}_t = \mathcal{F}_t \vee \sigma(L_{T_\Upsilon}) \tag{5.12}$$

Thus, the information yield is given by Equation 5.1 and stems directly from Itō's theorem, i.e.

$$\mu_s^{\mathcal{G}} = \frac{\mathbb{E}[L_{T_\Upsilon} - L_s \mid \mathcal{G}_t]}{T_\Upsilon - s}$$

We commence with the information premium under the measure $\mathbb{P}$. First though, we need an auxiliary result (this is Proposition A.3 from Benth and Meyer-Brandis (2009)) which enables to move parts of the information yield from the integral.

Lemma 5.3.1. Transformation of information yield (Lévy case). *Let the* information yield *be given by Equation 5.1. Then the following identity holds:*

$$\mathbb{E}[L_{T_\Upsilon} - L_s \mid \mathcal{G}_t] = \frac{T_\Upsilon - s}{T_\Upsilon - t}\, \mathbb{E}[L_{T_\Upsilon} - L_t \mid \mathcal{G}_t]$$

where $t \leq s \leq T_\Upsilon$.

Proof. Note that this is the framework of Theorem 2.2.5 with $g(t) = \frac{1}{T_\Upsilon - t}$ and $f(t) = 1$. The resulting integral equation from the proof of that theorem is

$$\xi(s) - \xi(t) = \int_t^s \xi'(u)\; du = -\int_t^s \frac{1}{T_\Upsilon - u}\,\xi(u)\; du$$

$$\xi'(u) = -\frac{1}{T_\Upsilon - u}\,\xi(u)$$

This differential equation has solution

$$\xi(s) = \xi(t) e^{-\int_s^t \frac{1}{T_\Upsilon - v} dv} = \xi(t) e^{\log(T_\Upsilon - s) - \log(T_\Upsilon - t)} = \xi(t)\frac{T_\Upsilon - s}{T_\Upsilon - t}$$

and this is the required result. □

5.3.1.1. Real-world measure

We now have all ingredients to postulate:

Proposition 5.3.1. Information premium with delivery period (Lévy case). *Let $0 \le t \le T_1 < T_2 \le T_\Upsilon$ and assume the market filtration satisfies Equation 5.12. Then the information premium under $\mathbb{P}$ with delivery period is given by*

$$I_\mathcal{G}^\mathbb{P}(t, T_1, T_2; T_\Upsilon) = \frac{1}{T_2 - T_1}\hat{\beta}(t, T_1, T_2)\left(\frac{\mathbb{E}[L_{T_\Upsilon} - L_t|\mathcal{G}_t]}{T_\Upsilon - t} - \psi'_{L_1}(0)\right)$$

where $\hat{\beta}(\cdot)$ is defined as in Notation 3.4.1.

Proof. This is similar to the proof of Proposition 4.3.1 as provided by Benth and Meyer-Brandis (2009). Again, most terms cancel as filtrations coincide:

$$\begin{aligned} I_\mathcal{G}^\mathbb{P}(t, T_1, T_2; T_\Upsilon) &= \mathbb{E}\left[\frac{1}{T_2 - T_1}\int_{T_1}^{T_2} S(u)du|\mathcal{G}_t\right] - \mathbb{E}\left[\frac{1}{T_2 - T_1}\int_{T_1}^{T_2} S(u)du|\mathcal{F}_t\right] \\ &= \frac{1}{T_2 - T_1}\left(\mathbb{E}\left[\int_{T_1}^{T_2}\int_t^u e^{-\beta(u-s)}dL(s)du|\mathcal{G}_t\right] - \hat{\beta}(t, T_1, T_2)\phi'(0)\right) \end{aligned}$$

where we have used the $\mathcal{F}_t$-forward result (cf. Proposition 3.4.2) for the second term. Next, we plug in the information yield:

$$\mathbb{E}\left[\int_{T_1}^{T_2}\int_t^u e^{-\beta(u-s)}dL(s)du|\mathcal{G}_t\right] = \int_{T_1}^{T_2}\int_t^u e^{-\beta(u-s)}\frac{\mathbb{E}[L_{T_\Upsilon} - L_s|\mathcal{G}_t]}{T_\Upsilon - s}\; ds\; du$$

We use Lemma 5.3.1 to remove the dependence on s from the expectation:

$$\begin{aligned} \ldots &= \mathbb{E}\left[\int_{T_1}^{T_2}\int_t^u \frac{e^{-\beta(u-s)}}{T_\Upsilon - s}\,\frac{T_\Upsilon - s}{T_\Upsilon - t}\mathbb{E}[L_{T_\Upsilon} - L_t|\mathcal{G}_t]\; ds\; du|\mathcal{G}_t\right] \\ &= \frac{\mathbb{E}[L_{T_\Upsilon} - L_t|\mathcal{G}_t]}{T_\Upsilon - t}\int_{T_1}^{T_2}\int_t^u e^{-\beta(u-s)}ds\; du \\ &= \frac{\mathbb{E}[L_{T_\Upsilon} - L_t|\mathcal{G}_t]}{T_\Upsilon - t}\,\frac{1}{\beta}\left(T_2 - T_1 + \frac{1}{\beta}\left(e^{-\beta(T_2 - t)} - e^{-\beta(T_1 - t)}\right)\right) \\ &= \frac{\mathbb{E}[L_{T_\Upsilon} - L_t|\mathcal{G}_t]}{T_\Upsilon - t}\,\hat{\beta}(t, T_1, T_2) \end{aligned}$$

Collecting terms yields the desired result. □

This also proves Proposition 4.3.1 by taking limits of the above result.

Lemma 5.3.2. Limit of information premium (Lévy case). *For $0 \leq t \leq T_1 < T_2 \leq T_\Upsilon$ the information premium with delivery period becomes that without delivery period as $T_2 \to T_1$.*

Proof. The last term (in brackets) of both results already coincides and does not feature either T_1 or T_2. Thus, it suffices to calculate

$$\begin{aligned}\lim_{T_2 \to T_1} \frac{1}{T_2 - T_1} \hat{\beta}(t, T_1, T_2) &= \lim_{T_2 \to T_1} \frac{\frac{1}{\beta}(T_2 - T_1 + \frac{1}{\beta}\ (e^{-\beta(T_2-t)} - e^{-\beta(T_1-t)}))}{T_2 - T_1} \\ &\overset{L'H}{=} \lim_{T_2 \to T_1} \left(\frac{1}{\beta} - \frac{1}{\beta} e^{-\beta(T_2-t)}\right) \\ &= \frac{1}{\beta}\left(1 - e^{-\beta(T_1-t)}\right)\end{aligned}$$

renaming $T_1 = T$, this is exactly the factor from Proposition 4.3.1. □

For the case that $0 \leq t \leq T_1 < T_2 \leq T_\Upsilon$ one finds the following relation between the premia:

Corollary 5.3.1. Relation of premia (Lévy case). *The relation between the information premium with and without delivery period is given by*

$$I_{\mathcal{G}}^{\mathbb{P}}(t, T_1, T_2; T_\Upsilon) = I_{\mathcal{G}}^{\mathbb{P}}(t, T_1; T_\Upsilon) \frac{\hat{\beta}(t, T_1, T_2)}{\frac{1}{\beta}(T_2 - T_1)(1 - e^{-\beta(T_1-t)})}$$

If the future information known to the market is located on the time axis before the delivery period, we can use properties of the filtrations in much the same way as in Benth and Meyer-Brandis (2009, Proposition 3.2) to calculate the information premium:

Proposition 5.3.2. Information premium (information before delivery, Lévy case). *For $0 \leq t < T_\Upsilon \leq T_1 < T_2$ and market filtration as in Equation 5.12 the information premium is given by:*

$$I_{\mathcal{G}}^{\mathbb{P}}(t, T_1, T_2; T_\Upsilon) = \frac{\bar{\beta}(T_\Upsilon, T_1, T_2)}{\frac{1}{\beta}(T_2 - T_1)} \left(1 - e^{-\beta(T_\Upsilon - t)}\right) \left(\frac{\mathbb{E}[L_{T_\Upsilon} - L_t | \mathcal{G}_t]}{T_\Upsilon - t} - \psi'_{L_1}(0)\right)$$

where again $\bar{\beta}(\cdot)$ is defined as in Notation 3.4.1.

Proof. Seasonality and base components cancel. Then, we write Y_u in terms of T_Υ:

$$\begin{aligned}I_{\mathcal{G}}^{\mathbb{P}}(t, T_1, T_2; T_\Upsilon) &= \frac{1}{T_2 - T_1} \left(\mathbb{E}\left[\int_{T_1}^{T_2} Y_u du | \mathcal{G}_t\right] - \mathbb{E}\left[\int_{T_1}^{T_2} Y_u du | \mathcal{F}_t\right]\right) \\ &= \frac{1}{T_2 - T_1} \left(\mathbb{E}\left[\int_{T_1}^{T_2} \left(e^{-\beta(u - T_\Upsilon)} Y_{T_\Upsilon} + \int_{T_\Upsilon}^{u} e^{-\beta(u-v)} dL_v\right) du | \mathcal{G}_t\right]\right. \\ &\quad \left. - \mathbb{E}\left[\int_{T_1}^{T_2} \left(e^{-\beta(u - T_\Upsilon)} Y_{T_\Upsilon} + \int_{T_\Upsilon}^{u} e^{-\beta(u-v)} dL_v\right) du | \mathcal{F}_t\right]\right)\end{aligned}$$

Now, the filtrations satisfy $\mathcal{G}_t \subseteq \mathcal{F}_{T_\Upsilon}$ as well as $\mathcal{F}_t \subseteq \mathcal{F}_{T_\Upsilon}$ so that we make use of the tower property:

$$\begin{aligned}
\ldots &= \frac{1}{T_2 - T_1}\left(\mathbb{E}\left[\int_{T_1}^{T_2}\left(e^{-\beta(u-T_\Upsilon)}Y_{T_\Upsilon} + \mathbb{E}\left[\int_{T_\Upsilon}^{u} e^{-\beta(u-v)}dL_v|\mathcal{F}_{T_\Upsilon}\right]\right)du|\mathcal{G}_t\right]\right.\\
&\qquad \left.- \mathbb{E}\left[\int_{T_1}^{T_2}\left(e^{-\beta(u-T_\Upsilon)}Y_{T_\Upsilon} + \mathbb{E}\left[\int_{T_\Upsilon}^{u} e^{-\beta(u-v)}dL_v|\mathcal{F}_{T_\Upsilon}\right]\right)du|\mathcal{F}_t\right]\right)\\
&= \frac{1}{T_2 - T_1}\left(\int_{T_1}^{T_2} e^{-\beta(u-T_\Upsilon)}du\ \mathbb{E}[Y_{T_\Upsilon}|\mathcal{G}_t] + \mathbb{E}\left[\hat{\beta}(T_\Upsilon, T_1, T_2)\mathbb{E}[L_1]|\mathcal{G}_t\right]\right.\\
&\qquad \left.- \int_{T_1}^{T_2} e^{-\beta(u-T_\Upsilon)}du\ \mathbb{E}[Y_{T_\Upsilon}|\mathcal{F}_t] - \mathbb{E}\left[\hat{\beta}(T_\Upsilon, T_1, T_2)\mathbb{E}[L_1]|\mathcal{F}_t\right]\right)\\
&= \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} e^{-\beta(u-T_\Upsilon)}du\ \left(\mathbb{E}[Y_{T_\Upsilon}|\mathcal{G}_t] - \mathbb{E}[Y_{T_\Upsilon}|\mathcal{F}_t]\right)
\end{aligned}$$

where we have used the fact that the inner expectation is just a number. We can write this in terms of the information premium without delivery period.

$$I_{\mathcal{G}}^{\mathbb{P}}(t, T_1, T_2; T_\Upsilon) = \frac{1}{T_2 - T_1}\bar{\beta}(T_\Upsilon, T_1, T_2)\ I_{\mathcal{G}}^{\mathbb{P}}(t, T_\Upsilon; T_\Upsilon)$$

which yields the result. □

We can now formulate the omitted proof of the corresponding delivery time result from Section 4.3, again, as the limit behaviour of the result above.

Lemma 5.3.3. Limit of information premium (before delivery, Lévy case). *For $0 \le t < T_\Upsilon \le T_1 < T_2$ and $T_2 \to T_1$ the information premium takes the form of Corollary 4.3.1.*

Proof. We take limits of terms depending on T_1, T_2 of the result of Proposition 5.3.2

$$\begin{aligned}
\lim_{T_2 \to T_1} \frac{\bar{\beta}(T_\Upsilon, T_1, T_2)}{\frac{1}{\beta}(T_2 - T_1)} &= \lim_{T_2 \to T_1} \frac{1}{T_2 - T_1}\bar{\beta}(T_\Upsilon, T_1, T_2)\\
&\overset{L'H}{=} \lim_{T_2 \to T_1} \frac{1}{\beta}\ \left(-\beta e^{-\beta(T_1 - T_\Upsilon)}\right)\\
&= e^{-\beta(T_1 - T_\Upsilon)}
\end{aligned}$$

which is the anticipated result. □

In practice, the most relevant case will be if the future information of the market filtration is located on the time axis somewhere between T_1 and T_2. It turns out that the information premium in this case can be expressed as a linear combination of Proposition 5.3.1 and Proposition 5.3.2.

Corollary 5.3.2. Information premium (information during delivery, Lévy case). *For $0 \le t \le T_1 < T_\Upsilon < T_2$ and the market filtration specified by Equation 5.12 the information premium is given by:*

$$I_{\mathcal{G}}^{\mathbb{P}}(t, T_1, T_2; T_\Upsilon) = \frac{T_\Upsilon - T_1}{T_2 - T_1}\underbrace{I_{\mathcal{G}}^{\mathbb{P}}(t, T_1, T_\Upsilon; T_\Upsilon)}_{\textit{Proposition } 5.3.1} + \frac{T_2 - T_\Upsilon}{T_2 - T_1}\underbrace{I_{\mathcal{G}}^{\mathbb{P}}(t, T_\Upsilon, T_2; T_\Upsilon)}_{\textit{Proposition } 5.3.2}$$

Proof. This is simply splitting the delivery period integral into two parts and adjusting for factors. □

There are, of course, two more possible setups for the time axis, namely those for which t is during the delivery period.

Corollary 5.3.3. Information premium during delivery period (Lévy case). *Let the market filtrations be given by Equation 5.12. Then:*

1. *For the case $0 \leq T_1 < t \leq T_2 \leq T_\Upsilon$ the information premium is given by:*

$$I_{\mathcal{G}}^{\mathbb{P}}(t,T_1,T_2;T_\Upsilon) = \frac{1}{T_2-T_1}(T_2-t)\underbrace{I_{\mathcal{G}}^{\mathbb{P}}(t,t,T_2;T_\Upsilon)}_{Proposition\ 5.3.1}$$

2. *Furthermore, for $0 \leq T_1 < t < T_\Upsilon < T_2$ it is:*

$$I_{\mathcal{G}}^{\mathbb{P}}(t,T_1,T_2;T_\Upsilon) = \frac{T_\Upsilon - t}{T_2-T_1}\underbrace{I_{\mathcal{G}}^{\mathbb{P}}(t,t,T_\Upsilon;T_\Upsilon)}_{Proposition\ 5.3.1} + \frac{T_2-T_\Upsilon}{T_2-T_1}\underbrace{I_{\mathcal{G}}^{\mathbb{P}}(t,T_\Upsilon,T_2;T_\Upsilon)}_{Proposition\ 5.3.2}$$

Proof. For the interval $[T_1,t]$ market and historical filtration coincide and, thus, these parts of the integrals cancel. Hence, we can confine our attention to the new delivery period $[t,T_2]$. □

5.3.1.2. Risk-neutral measure

Finding an expression for the information premium after a measure change as conducted in Section 3.4.2 does not require additional techniques. We can apply results from the last chapter and propose:

Proposition 5.3.3. Information premium (risk-neutral, Lévy case). *Under the parametric measure $\mathbb{Q}$, $0 \leq t \leq T_1 < T_2 \leq T_\Upsilon$ and market filtration as given by Equation 5.12 the information premium is given by:*

$$\begin{aligned} &I_{\mathcal{G}}^{\mathbb{Q}}(t,T_1,T_2;T_\Upsilon)\\ &\quad= \frac{1}{T_2-T_1}\left(\hat{\beta}(t,T_1,T_2)\frac{\mathbb{E}^{\mathbb{Q}}[L_{T_\Upsilon}-L_t|\mathcal{G}_t]}{T_\Upsilon - t} - \int_t^{T_2}\psi'_{L_1}(\theta_L(s))\bar{\beta}(s,T_1,T_2)ds\right)\end{aligned}$$

Proof. We have calculated the risk-neutral forward price in Proposition 3.4.3, hence we concentrate on the expectation under the market filtration:

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}\left[\int_{T_1}^{T_2}\int_t^u e^{-\beta(u-s)}dL_s du \mid \mathcal{G}_t\right]\\ &\quad= \mathbb{E}^{\mathbb{Q}}\left[\int_{T_1}^{T_2}\int_t^u e^{-\beta(u-s)}d\left(\xi_s + \int_0^s \frac{\mathbb{E}^{\mathbb{Q}}[L_{T_\Upsilon}-L_v \mid \mathcal{G}_s]}{T_\Upsilon - v}dv\right)du \mid \mathcal{G}_t\right]\\ &\quad= \int_{T_1}^{T_2}\int_t^u e^{-\beta(u-s)}\frac{\mathbb{E}^{\mathbb{Q}}[L_{T_\Upsilon}-L_s \mid \mathcal{G}_t]}{T_\Upsilon - s}dsdu\end{aligned}$$

This is because L_t was an $(\mathcal{F},\mathbb{P})$-Lévy process and ξ_t a $(\mathcal{G},\mathbb{Q})$-martingale. Itō's theorem as well as auxiliary result Theorem 2.2.5 obviously work for any equivalent measure $\mathbb{Q}$. Solving integrals and collecting terms yields the result. □

We remark that the technique we apply is the initial enlargement of filtrations. The market filtration provides, at t, a value for either the Lévy process at T_Υ or at least its expectation. Now, both measures $\mathbb{P}$ and $\mathbb{Q}$ coincide at t, and thus, so does the expectation of L_{T_Υ} under both measures.

Clearly, for all other arrangements of the time axis, the information premium can be found along the same lines as in Proposition 5.3.3 and the preceding section. These cases are left out for brevity.

5.3.2. Information Premium with Base Component Information

Now, we will consider the framework of Section 5.2.3 meaning that we will assume the market filtration to be given by:

$$\mathcal{G}_t \subseteq \mathcal{H}_t = \mathcal{F}_t \vee \sigma(X_{T_\Upsilon}) \tag{5.13}$$

Thus, the information yield is given by Proposition 5.2.1 or Corollary 5.2.2:

$$\mu_t^{\mathcal{G}} = \left(\int_t^{T_\Upsilon} e^{\alpha u} dW_u \right) \frac{2\alpha e^{\alpha t}}{e^{2\alpha T_\Upsilon} - e^{2\alpha t}} = \frac{1}{\sigma} e^{\alpha T_\Upsilon} \left(X_{T_\Upsilon} - e^{-\alpha(T_\Upsilon - t)} X_t \right) a(t)$$

For readability, we will also use the auxiliary function $a(t)$ as in Equation 5.10. Again, we need the auxiliary result:

Lemma 5.3.4. Transformation of information yield (base component). *Let the* information yield *be given by Proposition 5.2.1. Then:*

$$\mathbb{E}\left[\int_s^{T_\Upsilon} e^{\alpha u} dW_u \mid \mathcal{G}_t \right] = \frac{e^{2\alpha T_\Upsilon} - e^{2\alpha s}}{e^{2\alpha T_\Upsilon} - e^{2\alpha t}} \, \mathbb{E}\left[\int_t^{T_\Upsilon} e^{\alpha u} dW_u \mid \mathcal{G}_t \right]$$

where $t \leq s \leq T_\Upsilon$.

Proof. This is again Theorem 2.2.5 with $g(t) = a(t)$ as well as $f(t) = e^{\alpha t}$. The corresponding integral equation is:

$$\int_t^s \xi'(u) du = - \int_t^s e^{\alpha u} \; a(u) \; \xi(u) \; du$$

which has solution

$$\xi(s) = \xi(t) \exp\left(- \int_t^s e^{\alpha u} a(u) du \right)$$

and the integral in the exponent can be calculated using substitution (for example with $x = e^{2\alpha u}$, $dx = 2\alpha^{2\alpha u} du \Leftrightarrow du = \frac{1}{2\alpha}\frac{1}{x} dx$) and the solution is:

$$- \int_t^s e^{\alpha u} \frac{2\alpha e^{\alpha u}}{e^{2\alpha T_\Upsilon} - e^{2\alpha u}} du = \log\left(e^{2\alpha T_\Upsilon} - e^{2\alpha s} \right) - \log\left(e^{2\alpha T_\Upsilon} - e^{2\alpha t} \right)$$

Bringing together terms yields the result. $\square$

As in the preceding section, we need to consider different scenarios for the time axis. Again, we start with the case that extra information is provided at some date after the delivery period.

Proposition 5.3.4. Information premium with delivery period (base component). *Let the market filtration be given by Equation 5.13. Furthermore, let $0 \le t \le T_1 < T_2 \le T_\Upsilon$. Then the information premium under the real-world measure with delivery period is given by*

$$I^{\mathbb{P}}_{\mathcal{G}}(t,T_1,T_2;T_\Upsilon) = \frac{1}{T_2-T_1}\frac{1}{\alpha}\left(\frac{e^{2\alpha T_2}+e^{2\alpha t}}{e^{\alpha T_2}} - \frac{e^{2\alpha T_1}+e^{2\alpha t}}{e^{\alpha T_1}}\right)\frac{e^{\alpha T_\Upsilon}\mathbb{E}[X_{T_\Upsilon}|\mathcal{G}_t] - e^{\alpha t}X_t}{e^{2\alpha T_\Upsilon}-e^{2\alpha t}}$$

Proof. We use the same techniques as in the proof of Proposition 5.3.1:

$$\begin{aligned} I^{\mathbb{P}}_{\mathcal{G}}(t,T_1,T_2;T_\Upsilon) &= \frac{1}{T_2-T_1}\mathbb{E}\left[\int_{T_1}^{T_2}\int_t^u \sigma e^{-\alpha(u-s)}dW_s du|\mathcal{G}_t\right] \\ &= \frac{\sigma}{T_2-T_1}\mathbb{E}\left[\int_{T_1}^{T_2}\int_t^u e^{-\alpha(u-s)}\left(a(s)\mathbb{E}\left[\int_s^{T_\Upsilon} e^{av}dW_v|\mathcal{G}_s\right]\right)dsdu|\mathcal{G}_t\right] \\ &= \frac{\sigma}{T_2-T_1}\int_{T_1}^{T_2}\int_t^u e^{-\alpha(u-s)}a(s)\mathbb{E}\left[\int_s^{T_\Upsilon} e^{av}dW_v|\mathcal{G}_t\right]dsdu \end{aligned}$$

Now, we apply the auxiliary result Lemma 5.3.4 to simplify:

$$\begin{aligned} \ldots &= \frac{\sigma}{T_2-T_1}\int_{T_1}^{T_2}\int_t^u e^{-\alpha(u-s)}a(s)\frac{e^{2\alpha T_\Upsilon}-e^{2\alpha s}}{e^{2\alpha T_\Upsilon}-e^{2\alpha t}}\,\mathbb{E}\left[\int_t^{T_\Upsilon} e^{av}dW_v \mid \mathcal{G}_t\right]dsdu \\ &= \frac{\sigma}{T_2-T_1}\,\frac{\mathbb{E}\left[\int_t^{T_\Upsilon} e^{av}dW_v|\mathcal{G}_t\right]}{e^{2\alpha T_\Upsilon}-e^{2\alpha t}}\int_{T_1}^{T_2}\int_t^u 2\alpha e^{-\alpha u}e^{2\alpha s}dsdu \\ &= \frac{1}{T_2-T_1}\,\frac{e^{\alpha T_\Upsilon}\mathbb{E}\left[X_{T_\Upsilon}|\mathcal{G}_t\right]-e^{\alpha t}X_t}{e^{2\alpha T_\Upsilon}-e^{2\alpha t}}\int_{T_1}^{T_2} e^{\alpha u}-e^{2\alpha t}e^{-\alpha u}du \end{aligned}$$

where we have used the representation in terms of the base component (cf. Corollary 5.2.1). Evaluating the last integral yields the result. ☐

We find the information premium for extra information before the delivery period using similar methods as before:

Proposition 5.3.5. Information premium (information before delivery, base case). *For $0 \le t < T_\Upsilon \le T_1 < T_2$ and market filtration specified by Equation 5.13 the information premium takes the following form:*

$$I^{\mathbb{P}}_{\mathcal{G}}(t,T_1,T_2;T_\Upsilon) = \frac{1}{T_2-T_1}\;\bar{\alpha}(T_\Upsilon,T_1,T_2)\left(\mathbb{E}\left[X_{T_\Upsilon}\mid\mathcal{G}_t\right]-e^{-\alpha(T_\Upsilon-t)}X_t\right)$$

Proof. As in the proof of Proposition 5.3.2 we make use of the tower property:

$$\begin{aligned} \mathbb{E}\left[\int_{T_1}^{T_2}X_u du|\mathcal{G}_t\right] &= \mathbb{E}\left[\int_{T_1}^{T_2}\left(e^{-\alpha(u-T_\Upsilon)}X_{T_\Upsilon}+\int_{T_\Upsilon}^u e^{-\alpha(u-s)}dW_s\right)du|\mathcal{G}_t\right] \\ &= \mathbb{E}\left[\int_{T_1}^{T_2}\left(e^{-\alpha(u-T_\Upsilon)}X_{T_\Upsilon}+\mathbb{E}\left[\int_{T_\Upsilon}^u e^{-\alpha(u-s)}dW_s|\mathcal{F}_{T_\upsilon}\right]\right)du|\mathcal{G}_t\right] \\ &= \bar{\alpha}(T_\Upsilon,T_1,T_2)\mathbb{E}\left[X_{T_\Upsilon}\mid\mathcal{G}_t\right] \end{aligned}$$

Plugging this expression into the definition of the information premium proves the proposition. □

The following lemma will provide the hitherto omitted proofs for the delivery-time results from Section 4.3:

Lemma 5.3.5. Limit of information premium (base case). *For both $T_\Upsilon \leq T_1$ as well as for $T_\Upsilon \geq T_2$, the limit of the information premium as $T_2 \to T_1$ takes the form of Proposition 4.3.2 and Corollary 4.3.2 correspondingly.*

Proof. We need to evaluate limits of the expressions including T_2. This is a straightforward calculation, again with an application of L'Hospital's rule. For $T_\Upsilon \geq T_2$ we have:

$$\lim_{T_2 \to T_1} \frac{1}{T_2 - T_1} \left(\frac{e^{2\alpha T_2} + e^{2\alpha t}}{e^{\alpha T_2}} - \frac{e^{2\alpha T_1} + e^{2\alpha t}}{e^{\alpha T_1}} \right) \overset{L'H}{=} \alpha e^{\alpha T_1} \left(e^{2\alpha T_1} - e^{2\alpha t} \right)$$

whereas for $T_\Upsilon \leq T_1$:

$$\lim_{T_2 \to T_1} \frac{\bar{\alpha}(T_\Upsilon, T_1, T_2)}{T_2 - T_1} \overset{L'H}{=} e^{-\alpha(T_1 - T_\Upsilon)}$$

Renaming $T_1 = T$ and bringing together terms yields exactly the results from Chapter 4. □

Remembering Corollary 5.3.2 as well as Corollary 5.3.3 from last section we trivially find the remaining cases:

Corollary 5.3.4. Information premium (information during delivery, base case). *For $0 \leq t \leq T_1 < T_\Upsilon < T_2$ and the market filtration specified by Equation 5.13 the information premium is given by:*

$$I_{\mathcal{G}}^{\mathbb{P}}(t, T_1, T_2; T_\Upsilon) = \frac{T_\Upsilon - T_1}{T_2 - T_1} \underbrace{I_{\mathcal{G}}^{\mathbb{P}}(t, T_1, T_\Upsilon; T_\Upsilon)}_{Proposition\ 5.3.4} + \frac{T_2 - T_\Upsilon}{T_2 - T_1} \underbrace{I_{\mathcal{G}}^{\mathbb{P}}(t, T_\Upsilon, T_2; T_\Upsilon)}_{Proposition\ 5.3.5}$$

Corollary 5.3.5. Information premium during delivery period (base case). *Let the market filtrations be given by Equation 5.13. Then:*

1. *For the case $0 \leq T_1 < t \leq T_2 \leq T_\Upsilon$ the information premium is given by:*

$$I_{\mathcal{G}}^{\mathbb{P}}(t, T_1, T_2; T_\Upsilon) = \frac{1}{T_2 - T_1} (T_2 - t) \underbrace{I_{\mathcal{G}}^{\mathbb{P}}(t, t, T_2; T_\Upsilon)}_{Proposition\ 5.3.4}$$

2. *Furthermore, for $0 \leq T_1 < t < T_\Upsilon < T_2$ it is:*

$$I_{\mathcal{G}}^{\mathbb{P}}(t, T_1, T_2; T_\Upsilon) = \frac{T_\Upsilon - t}{T_2 - T_1} \underbrace{I_{\mathcal{G}}^{\mathbb{P}}(t, t, T_\Upsilon; T_\Upsilon)}_{Proposition\ 5.3.4} + \frac{T_2 - T_\Upsilon}{T_2 - T_1} \underbrace{I_{\mathcal{G}}^{\mathbb{P}}(t, T_\Upsilon, T_2; T_\Upsilon)}_{Proposition\ 5.3.5}$$

With the formulae found in this section and remembering the discussion in Section 5.3.1.2 it is now trivial to identify the information premium under a risk-neutral measure. We will leave out these results for the sake of brevity.

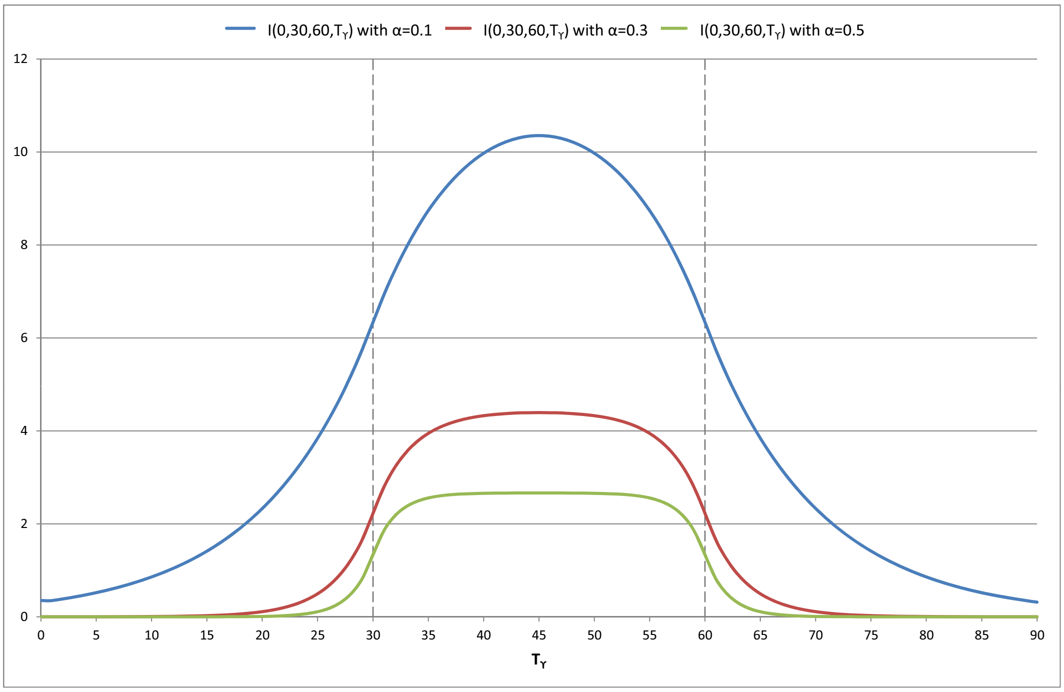


Figure 5.3.1.: Stylised information premium: Moving information. In $t = 0$ with delivery in $[30, 60]$, $X_0 = 0$, $\mathbb{E}[X_{T_\Upsilon} \mid \mathcal{G}_0] = 20$ and different mean reversion.

5.3.3. Stylised Examples and Discussion

In order to provide some intuition for the formulae found in this section we will now have a brief look at some stylised examples. The setup will be as follows: we will assume that today is $t = 0$ and we are considering a forward contract with delivery from day $T_1 = 30$ until $T_2 = 60$. The market is given additional information that the base component of the spot at T_Υ will be an estimated 20 €, i.e. $\mathbb{E}[X_{T_\Upsilon} \mid \mathcal{G}_0] = 20$.

Figure 5.3.1 illustrates the value of the formulae from Section 5.3.2 for different values of T_Υ (x-axis) given that the base component today has zero value, i.e. $X_0 = 0$ as well as for different speeds of mean reversion parameters. All three graphs show that the information premium takes its largest values for a T_Υ that is located during the delivery period whereas the value of the information premium tends to zero if the extra information is far before or after the delivery. Clearly, the greater α the lesser will be the significance of the future information. Indeed, the three graphs do not only take lesser values for larger α (around 10 € for $\alpha = 0.1$, around 4 € for $\alpha = 0.3$ and around 2.50 € for $\alpha = 0.5$) but they also become more angular around T_1 and T_2 so that extra information with large speed of mean-reversion has hardly any monetary value if not during delivery. Furthermore, visual examination suggests that the maximum value of the information premium is attained for $T_\Upsilon = 45$, i.e. right in the middle of the delivery period. Rewriting the result of Corollary 5.3.4, though, we find that it is a polynomial of degree seven in terms of $e^{\alpha T_\Upsilon}$. It is well known (by the *Abel-Ruffini theorem*) that the roots of its derivative cannot be found

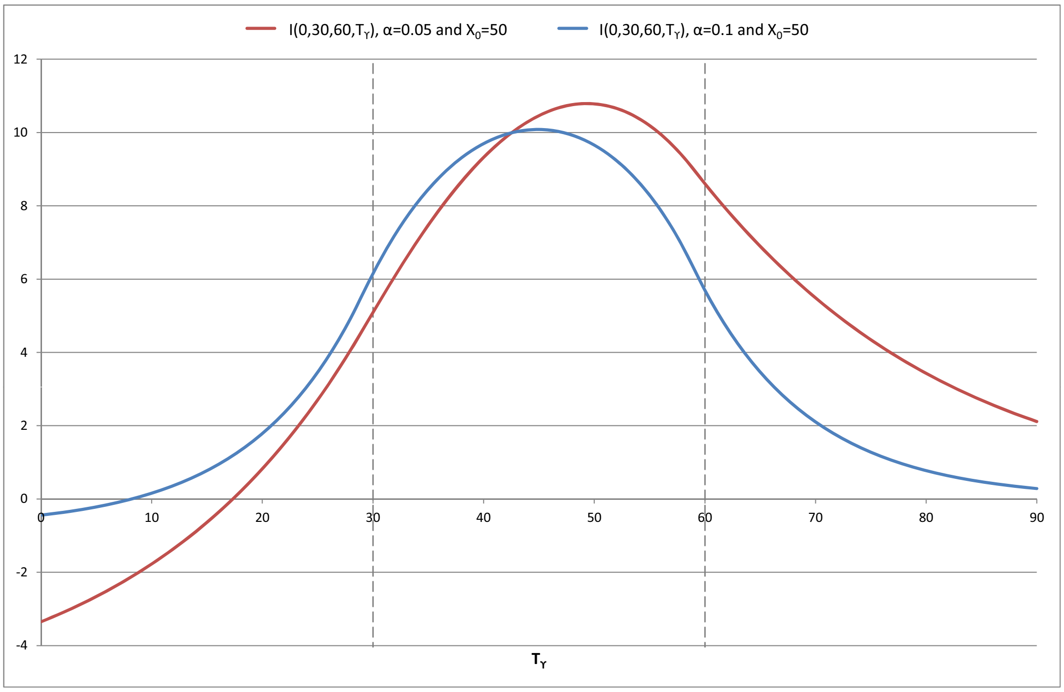


Figure 5.3.2.: Stylised information premium: Different start values. In $t = 0$ with delivery in $[30, 60]$, $X_0 = 50$, $\mathbb{E}[X_{T_\Upsilon} \mid \mathcal{G}_0] = 20$ and different mean reversion.

analytically. Still, if we set $t = 0$ as well as $X_0 = 0$, the derivative of the information premium between T_1 and T_2 will reduce to a polynomial of degree four in terms of $e^{\alpha T_\Upsilon}$. To see this, we first plug in $t = 0$ and X_0 and examine the simplified version of the result of Corollary 5.3.4:

$$I^{\mathbb{P}}_{\mathcal{G}}(0, T_1, T_2; T_\Upsilon) = \frac{\mathbb{E}[X_{T_\Upsilon} \mid \mathcal{G}_t]}{\alpha(T_2 - T_1)} \left(\frac{e^{2\alpha T_\Upsilon} + 1}{e^{2\alpha T_\Upsilon} - 1} - \frac{(e^{2\alpha T_1} + 1)e^{\alpha T_\Upsilon}}{e^{\alpha T_1}(e^{2\alpha T_\Upsilon} - 1)} - \frac{e^{\alpha T_\Upsilon}}{e^{\alpha T_2}} + 1 \right) \quad (5.14)$$

Taking the derivative of Equation 5.14 in T_Υ, equalising with zero and conducting some further calculations results in:

$$0 \stackrel{!}{=} \frac{\partial I^{\mathbb{P}}_{\mathcal{G}}(0, T_1, T_2; T_\Upsilon)}{\partial T_\Upsilon}$$

$$0 = e^{4\alpha T_\Upsilon} + \left(\frac{e^{2\alpha T_1} e^{\alpha T_2} + e^{\alpha T_2}}{e^{\alpha T_1}} + 2 \right) e^{2\alpha T_\Upsilon} - 4e^{\alpha T_1} e^{\alpha T_\Upsilon} + \left(\frac{e^{2\alpha T_1} e^{\alpha T_2} + e^{\alpha T_2}}{e^{\alpha T_1}} - 1 \right)$$

which is a quartic polynomial in terms of $e^{\alpha T_\Upsilon}$. This can be solved using for example the method of *Lodovico Ferrari*. The resulting solution (calculated by a computer algebra software) is enormously complicated but takes exactly the value $T_\Upsilon = 45$ confirming both intuition and visual impression. We also remark the independence of the value of the future information of the solution.

Figure 5.3.2 illustrates that the symmetry in the location of T_Υ apparent in Figure 5.3.1 disappears if the start value X_0 is altered. Here, it is set to 50 €. We see that for a small α not only does the information premium take its largest value closer towards T_2 but we experience a negative premium for extra information shortly after $t = 0$. The reason is that the value of the $\mathcal{F}$-forward depends on X_0 which is very large. Extra information that the spot will decline to 20 € faster than anticipated by the historical filtration will consequently result in a negative premium.

5.4. Multiple Pieces of Future Information

So far, in this chapter, we have considered some single time T_Υ at which the market has access to additional information. Now, the goal of this section is to find a general expression for the information premium given some $n \in \mathbb{Z}^+$ pieces of future information at time points $T_{\Upsilon_1} < T_{\Upsilon_2} < \dots T_{\Upsilon_n}$. For closed-form solutions it will be necessary to accept the following restrictions: we will assume that the market filtration equals the filtration with precise future information, i.e. $\mathcal{G}_t = \mathcal{H}_t$. Also, we will concentrate on the case of base component information. Thus, the market filtration in this section will be:

$$\mathcal{G}_t = \mathcal{H}_t = \mathcal{F}_t \vee \sigma(X_{T_{\Upsilon_1}}, X_{T_{\Upsilon_2}}, \dots, X_{T_{\Upsilon_n}}) \tag{5.15}$$

Similar to Benth and Meyer-Brandis (2009, page 133) we will use:

Notation 5.4.1. Subfiltrations. We define the following filtrations which are coarser than the market filtration $\mathcal{H}_t$:

$$\begin{aligned}\mathcal{H}_t^{\Upsilon_i} &= \mathcal{F}_t \vee \sigma\left(S_{T_{\Upsilon_i}}\right)\\ \mathcal{H}_t^{\Upsilon_i,\Upsilon_j} &= \mathcal{F}_t \vee \sigma\left(S_{T_{\Upsilon_i}}, S_{T_{\Upsilon_j}}\right)\end{aligned}$$

where i, j satisfy $1 \leq i < j \leq n$.

The reason why we need the market filtration to include complete information is that we will make use of the relation

$$\mathcal{H}_t^{\Upsilon_1,\Upsilon_2} = \mathcal{H}_{T_{\Upsilon_1}}^{\Upsilon_2} \text{ on } [T_{\Upsilon_1}, T_{\Upsilon_2}] \tag{5.16}$$

Hence, this relation allows to reduce the number of pieces of additional information that need to be considered in a given time interval.

Furthermore, in order to identify an expression for the information premium, we will consider the case $n = 2$, i.e. information at two future time points, first. There are two reasons for this approach: on the one hand, we will be able to use the results for different arrangements of two pieces of information as building blocks for the general framework. On the other hand, the general result will be quite complicated and the case of two pieces of information will help to both understand the techniques applied and to illustrate them.

5.4.1. Two Pieces of Future Information

Just as with one piece of future information we find different results for different arrangements of the time axis. Before we discuss these cases, we commence with a simple lemma, which will be useful later. This lemma enables to write expressions featuring delivery periods in terms of the information premium without a delivery period (i.e. Proposition 4.3.2 and Corollary 4.3.2). We have seen a similar idea in the last line of the proof of Proposition 5.3.2.

Lemma 5.4.1. Auxiliary result. *For $t \leq T_{\Upsilon_1} \leq T_1 \leq T_2$ one has the identity*

$$\mathbb{E}\left[\int_{T_1}^{T_2}\int_t^{T_{\Upsilon_1}} e^{-\alpha(u-s)}dW_s du | \mathcal{H}_t^{\Upsilon_1}\right] = \frac{1}{\sigma}\bar{\alpha}(T_{\Upsilon_1}, T_1, T_2) I_{\mathcal{H}^{\Upsilon_1}}(t, T_{\Upsilon_1}; T_{\Upsilon_1})$$

Proof. After moving the expectation into the integrals we calculate:

$$\begin{aligned}\int_{T_1}^{T_2} e^{\alpha T_{\Upsilon_1}} e^{-\alpha u}\mathbb{E}\left[\int_t^{T_{\Upsilon_1}} e^{-\alpha(T_{\Upsilon_1}-s)}dW_s \mid \mathcal{H}_t^{\Upsilon_1}\right] du &= \int_{T_1}^{T_2} e^{\alpha u}\frac{e^{-\alpha T_{\Upsilon_1}}}{\sigma} I_{\mathcal{H}^{\Upsilon_1}}(t, T_{\Upsilon_1})du \\ &= \frac{1}{\sigma} e^{\alpha T_{\Upsilon_1}} I_{\mathcal{H}^{\Upsilon_1}}(t, T_{\Upsilon_1}) \int_{T_1}^{T_2} e^{-\alpha u}du\end{aligned}$$

solving the last integral gives the result as required. □

Now, we will state the results for the information premium given the different setups of the time axis. We will first consider one piece of information before the delivery period:

Figure 5.4.3.: Multiple information: Case one. Time axis for Proposition 5.4.1.

Proposition 5.4.1. Information premium (two pieces of information, case one). *Considering the case $t \leq T_{\Upsilon_1} < T_1 \leq T_{\Upsilon_2} \leq T_2$, i.e. additional information before and during the delivery period (as illustrated in Figure 5.4.3) the information premium is given by:*

$$I_{\mathcal{H}}(t, T_1, T_2; T_{\Upsilon_1}, T_{\Upsilon_2}) = \frac{\bar{\alpha}(T_{\Upsilon_1}, T_1, T_2)}{T_2 - T_1} I_{\mathcal{H}^{\Upsilon_1}}(t, T_{\Upsilon_1}; T_{\Upsilon_1}) + I_{\mathcal{H}^{\Upsilon_2}}(T_{\Upsilon_1}, T_1, T_2; T_{\Upsilon_2})$$

where the first expression is calculated according to Proposition 4.3.2 and the second according to Corollary 5.3.4.

Proof. As usual, terms cancel and we get

$$I_{\mathcal{H}}(t, T_1, T_2; T_{\Upsilon_1}, T_{\Upsilon_2}) = \frac{\sigma}{T_2 - T_1}\mathbb{E}\left[\int_{T_1}^{T_2}\int_t^u e^{-\alpha(u-s)}dW_s du \mid \mathcal{H}_t\right]$$

Now, we can split the inner integral into two parts. Furthermore, on the interval $[t, T_{\Upsilon_1}]$ it suffices to consider only the first piece of additional information. Similarly, we use the finding of Equation 5.16 to replace filtration $\mathcal{H}_t$ by $\mathcal{H}_{T_{\Upsilon_1}}^{\Upsilon_2}$:

$$\begin{aligned}&\mathbb{E}\left[\int_{T_1}^{T_2}\int_t^{T_{\Upsilon_1}} e^{-\alpha(u-s)}dW_s du \mid \mathcal{H}_t^{\Upsilon_1}\right] + \mathbb{E}\left[\int_{T_1}^{T_2}\int_{T_{\Upsilon_1}}^u e^{-\alpha(u-s)}dW_s du \mid \mathcal{H}_{T_{\Upsilon_1}}^{\Upsilon_2}\right] \\ &= \frac{1}{\sigma}\bar{\alpha}(T_{\Upsilon_1}, T_1, T_2) I_{\mathcal{H}^{\Upsilon_1}}(t, T_{\Upsilon_1}; T_{\Upsilon_1}) + \frac{T_2 - T_1}{\sigma} I_{\mathcal{H}^{\Upsilon_2}}(T_{\Upsilon_1}, T_1, T_2; T_{\Upsilon_2})\end{aligned}$$

where the first expectation has been solved using Lemma 5.4.1 and the second expression is to be evaluated utilising Corollary 5.3.4. Multiplying by $\frac{\sigma}{T_2 - T_1}$ provides the desired result. $\square$

The next case, which we will consider, features both pieces of additional information during the delivery period as illustrated in Figure 5.4.4.

Figure 5.4.4.: Multiple information: Case two. Time axis for Proposition 5.4.2.

Proposition 5.4.2. Information premium (two pieces of information, case two). *Considering the situation $t \leq T_1 \leq T_{\Upsilon_1} < T_{\Upsilon_2} \leq T_2$ as illustrated in Figure 5.4.4 the information premium is given by*

$$I_{\mathcal{H}}(t, T_1, T_2; T_{\Upsilon_1}, T_{\Upsilon_2}) = \frac{1}{T_2 - T_1}\big((T_{\Upsilon_1} - T_1) I_{\mathcal{H}^{\Upsilon_1}}(t, T_1, T_{\Upsilon_1}; T_{\Upsilon_1}) \\ + \bar{\alpha}(T_{\Upsilon_1}, T_{\Upsilon_1}, T_2) I_{\mathcal{H}^{\Upsilon_1}}(t, T_{\Upsilon_1}; T_{\Upsilon_1})\big) + (T_2 - T_{\Upsilon_1}) I_{\mathcal{H}^{\Upsilon_2}}(T_{\Upsilon_1}, T_{\Upsilon_1}, T_2; T_{\Upsilon_2})$$

Proof. Ignoring factors and splitting both the inner and the outer integral we get three terms:

$$\begin{aligned} \ldots &= \mathbb{E}\left[\int_{T_1}^{T_{\Upsilon_1}} \int_t^u e^{-\alpha(u-s)} dW_s du \mid \mathcal{H}_t^{\Upsilon_1}\right] + \mathbb{E}\left[\int_{T_{\Upsilon_1}}^{T_2} \int_t^{T_{\Upsilon_1}} e^{-\alpha(u-s)} dW_s du \mid \mathcal{H}_t^{\Upsilon_1}\right] \\ &\quad + \mathbb{E}\left[\int_{T_{\Upsilon_1}}^{T_2} \int_{T_{\Upsilon_1}}^u e^{-\alpha(u-s)} dW_s du \mid \mathcal{H}_{T_{\Upsilon_1}}^{\Upsilon_2}\right] \\ &= \frac{1}{\sigma}(T_{\Upsilon_1} - T_1) I_{\mathcal{H}^{\Upsilon_1}}(t, T_1, T_{\Upsilon_1}; T_{\Upsilon_1}) + \frac{1}{\sigma}\frac{1}{\alpha}\left(1 - e^{-\alpha(T_2 - T_{\Upsilon_1})}\right) I_{\mathcal{H}^{\Upsilon_1}}(t, T_{\Upsilon_1}; T_{\Upsilon_1}) \\ &\quad + \frac{1}{\sigma}(T_2 - T_{\Upsilon_1}) I_{\mathcal{H}^{\Upsilon_2}}(T_{\Upsilon_1}, T_{\Upsilon_1}, T_2; T_{\Upsilon_2}) \end{aligned}$$

Here, Lemma 5.4.1 and case $t \geq T_1$ of Notation 3.4.1 have been used for the second term. The first term is to be calculated according to Proposition 5.3.4 and the third according to Corollary 5.3.4. Adjusting for factors then gives the result. $\square$

The third case, in which one piece of information is during the delivery period and one is thereafter takes almost the same form as Proposition 5.4.2:

Figure 5.4.5.: Multiple information: Case three. Time axis for Proposition 5.4.3.

Proposition 5.4.3. Information premium (two pieces of information, case three). *For the case $t \leq T_1 \leq T_{\Upsilon_1} < T_2 \leq T_{\Upsilon_2}$ which is illustrated in Figure 5.4.5, the information premium is again given by*

$$I_{\mathcal{H}}(t,T_1,T_2;T_{\Upsilon_1},T_{\Upsilon_2}) = \frac{1}{T_2 - T_1}\big((T_{\Upsilon_1} - T_1)I_{\mathcal{H}^{\Upsilon_1}}(t,T_1,T_{\Upsilon_1};T_{\Upsilon_1}) \\ + \bar{\alpha}(T_{\Upsilon_1},T_{\Upsilon_1},T_2)I_{\mathcal{H}^{\Upsilon_1}}(t,T_{\Upsilon_1};T_{\Upsilon_1})\big) + (T_2 - T_{\Upsilon_1})I_{\mathcal{H}^{\Upsilon_2}}(T_{\Upsilon_1},T_{\Upsilon_1},T_2;T_{\Upsilon_2})$$

Still, here, the third term needs to be calculated making use of Proposition 5.3.4 (i.e. additional information after delivery period) rather than Corollary 5.3.4 (corresponding to information during the delivery period).

Proof. See proof of Proposition 5.4.2. □

Figure 5.4.6.: Multiple information: Case four. Time axis for Proposition 5.4.4.

The last arrangement to be considered has one piece of information before and one after the delivery period.

Proposition 5.4.4. Information premium (two pieces of information, case four). *The information premium for additional knowledge both before and after the delivery period, i.e. for $t \leq T_{\Upsilon_1} < T_1 \leq T_2 \leq T_{\Upsilon_2}$ as illustrated in Figure 5.4.6 takes the form*

$$I_{\mathcal{H}}(t,T_1,T_2;T_{\Upsilon_1},T_{\Upsilon_2}) = \frac{\bar{\alpha}(T_{\Upsilon_1},T_1,T_2)}{T_2 - T_1}I_{\mathcal{H}^{\Upsilon_1}}(t,T_{\Upsilon_1};T_{\Upsilon_1}) + I_{\mathcal{H}^{\Upsilon_2}}(T_{\Upsilon_1},T_1,T_2;T_{\Upsilon_2})$$

Here, again, the first expression is calculated applying Proposition 4.3.2, the second applying Proposition 5.3.4.

Proof. Again, this is proved by splitting the inner integral and applying auxiliary Lemma 5.4.1. □

Now we are ready to state the general result with n pieces of future information added to the historical filtration.

5.4.2. The General Result

Of the n pieces of future information one will be situated before, one after and $n-2$ during the delivery period. The reason for this is that one can ignore any information after the delivery period following the one closest to it. Furthermore, we do not consider more pieces of information before the delivery. The setup of the time axis we are facing is illustrated in Figure 5.4.7.

As mentioned above, our approach will be to split integrals into elements for which only two pieces of future information need to be considered.

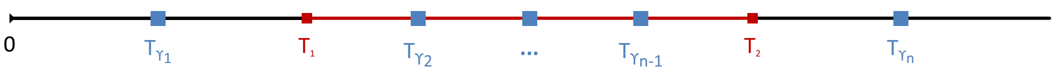


Figure 5.4.7.: Multiple information: General case. Time axis for Proposition 5.4.5.

We commence with the definition of the information premium:

$$I_{\mathcal{H}}(t, T_1, T_2; T_{\Upsilon_1}, \ldots, T_{\Upsilon_n}) = \frac{\sigma}{T_2 - T_1} \mathbb{E}\left[\int_{T_1}^{T_2} \int_t^u e^{-\alpha(u-s)} dW_s du \mid \mathcal{H}_t\right]$$

We decompose the expectation (ignoring factors) as follows:

$$\begin{aligned} \ldots = \mathbb{E}\left[\int_{T_1}^{T_{\Upsilon_2}} \int_t^u e^{-\alpha(u-s)} dW_s du \mid \mathcal{H}_t\right] + \mathbb{E}\left[\sum_{i=1}^{n-3} \int_{T_{\Upsilon_{i+1}}}^{T_{\Upsilon_{i+2}}} \int_t^u e^{-\alpha(u-s)} dW_s du \mid \mathcal{H}_t\right] \\ + \mathbb{E}\left[\int_{T_{\Upsilon_{n-1}}}^{T_2} \int_t^u e^{-\alpha(u-s)} dW_s du \mid \mathcal{H}_t\right] \end{aligned} \tag{5.17}$$

Now, the first of these three expressions can be evaluated using Proposition 5.4.1:

$$\mathbb{E}\left[\int_{T_1}^{T_{\Upsilon_2}} \int_t^u e^{-\alpha(u-s)} dW_s du | \mathcal{H}_t\right] = \frac{T_{\Upsilon_2} - T_1}{\sigma} I_{\mathcal{H}}^{\Upsilon_1, \Upsilon_2}(t, T_1, T_{\Upsilon_2}; T_{\Upsilon_1}, T_{\Upsilon_2}) \tag{5.18}$$

The latter two parts are more complicated. For the second term of Equation 5.17 we will, for now, only consider one i and make repeated use of auxiliary Lemma 5.4.1:

$$\begin{aligned} &\mathbb{E}\left[\int_{T_{\Upsilon_{i+1}}}^{T_{\Upsilon_{i+2}}} \int_t^u e^{-\alpha(u-s)} dW_s du \mid \mathcal{H}_t\right] \\ &\qquad = \mathbb{E}\left[\int_{T_{\Upsilon_{i+1}}}^{T_{\Upsilon_{i+2}}} \int_{T_{\Upsilon_{i+1}}}^u e^{-\alpha(u-s)} dW_s du \mid \mathcal{H}_{T_{\Upsilon_{i+1}}}^{\Upsilon_{i+2}}\right] \\ &\qquad + \sum_{j=1}^{i} \mathbb{E}\left[\int_{T_{\Upsilon_{i+1}}}^{T_{\Upsilon_{i+2}}} \int_{T_{\Upsilon_j}}^{T_{\Upsilon_{j+1}}} e^{-\alpha(u-s)} dW_s du \mid \mathcal{H}_{T_{\Upsilon_j}}^{\Upsilon_{j+1}}\right] \\ &\qquad + \mathbb{E}\left[\int_{T_{\Upsilon_{i+1}}}^{T_{\Upsilon_{i+2}}} \int_t^{T_{\Upsilon_1}} e^{-\alpha(u-s)} dW_s du \mid \mathcal{H}_t^{\Upsilon_1}\right] \end{aligned}$$

We can now continue by substituting

$$\begin{aligned}
\ldots =& \frac{1}{\sigma}\ (T_{\Upsilon_{i+2}} - T_{\Upsilon_{i+1}}) I_{\mathcal{H}^{\Upsilon_{i+1}}}(T_{\Upsilon_{i+1}}, T_{\Upsilon_{i+1}}, T_{\Upsilon_{i+2}}; T_{\Upsilon_{i+2}}) \\
&+ \sum_{j=1}^{i} \frac{1}{\sigma}\bar{\alpha}(T_{\Upsilon_{j+1}}, T_{\Upsilon_{i+1}}, T_{\Upsilon_{i+2}}) I_{\mathcal{H}^{\Upsilon_{j+1}}}(T_{\Upsilon_j}, T_{\Upsilon_{j+1}}; T_{\Upsilon_{j+1}}) \\
&+ \frac{1}{\sigma}\bar{\alpha}(T_{\Upsilon_1}, T_{\Upsilon_{i+1}}, T_{\Upsilon_{i+2}}) I_{\mathcal{H}^{\Upsilon_1}}(t, T_{\Upsilon_1}; T_{\Upsilon_1}) \\
=& \frac{1}{\sigma}\ (T_{\Upsilon_{i+2}} - T_{\Upsilon_{i+1}}) I_{\mathcal{H}^{\Upsilon_{i+1},\Upsilon_{i+2}}}(T_{\Upsilon_i}, T_{\Upsilon_{i+1}}, T_{\Upsilon_{i+2}}; T_{\Upsilon_{i+1}}, T_{\Upsilon_{i+2}}) && (5.19) \\
&+ \sum_{j=1}^{i-1} \frac{1}{\sigma}\bar{\alpha}(T_{\Upsilon_{j+1}}, T_{\Upsilon_{i+1}}, T_{\Upsilon_{i+2}}) I_{\mathcal{H}^{\Upsilon_{j+1}}}(T_{\Upsilon_j}, T_{\Upsilon_{j+1}}; T_{\Upsilon_{j+1}}) && (5.20) \\
&+ \frac{1}{\sigma}\bar{\alpha}(T_{\Upsilon_1}, T_{\Upsilon_{i+1}}, T_{\Upsilon_{i+2}}) I_{\mathcal{H}^{\Upsilon_1}}(t, T_{\Upsilon_1}; T_{\Upsilon_1}) && (5.21)
\end{aligned}$$

Here, in the last step, we have moved the last term of the sum into the first expression which allows using Proposition 5.4.1 (we remark that the first term of that proposition cancels because the left border of the delivery period equals the position of the first piece of additional information). We see that there are exactly $i+1$ terms added for each i in the sum. For the third expression of Equation 5.17 we can use similar techniques:

$$\begin{aligned}
&\mathbb{E}\left[\int_{T_{\Upsilon_{n-1}}}^{T_2} \int_t^u e^{-\alpha(u-s)} dW_s du \mid \mathcal{H}_t\right] \\
&\qquad = \frac{1}{\sigma}(T_2 - T_{\Upsilon_{n-1}}) I_{\mathcal{H}^{\Upsilon_{n-1},\Upsilon_n}}(T_{\Upsilon_{n-2}}, T_{\Upsilon_{n-1}}, T_2; T_{\Upsilon_{n-1}}, T_{\Upsilon_n}) && (5.22) \\
&\qquad + \sum_{j=1}^{n-3} \frac{1}{\sigma}\bar{\alpha}(T_{\Upsilon_{j+1}}, T_{\Upsilon_{n-1}}, T_2) I_{\mathcal{H}^{\Upsilon_{j+1}}}(T_{\Upsilon_j}, T_{\Upsilon_{j+1}}; T_{\Upsilon_{j+1}}) && (5.23) \\
&\qquad + \frac{1}{\sigma}\bar{\alpha}(T_{\Upsilon_1}, T_{\Upsilon_{n-1}}, T_2) I_{\mathcal{H}^{\Upsilon_1}}(t, T_{\Upsilon_1}; T_{\Upsilon_1}) && (5.24)
\end{aligned}$$

Summarising, we have now identified three terms involving the information premium with two pieces of information from Section 5.4.1: these are Equation 5.18, Equation 5.19 and Equation 5.22. We will now bring together terms and state the following proposition:

Proposition 5.4.5. Information premium (general case). *For the market filtration given by Equation 5.15 and the arrangement of the time axis as specified by Figure 5.4.7, i.e.* $t \leq T_{\Upsilon_1} < T_1 \leq T_{\Upsilon_2} \leq \ldots \leq T_{\Upsilon_{n-1}} \leq T_2 < T_{\Upsilon_n}$ *and a forward with* *delivery in $[T_1, T_2]$ the information premium is given by:*

$$
\begin{aligned}
&I_{\mathcal{H}}(t, T_1, T_2; T_{\Upsilon_1}, \ldots, T_{\Upsilon_n}) \\
&\quad = \frac{1}{T_2 - T_1}\Big((T_{\Upsilon_2} - T_1) I_{\mathcal{H}^{\Upsilon_1,\Upsilon_2}}(t, T_1, T_{\Upsilon_2}; T_{\Upsilon_1}, T_{\Upsilon_2}) \\
&\quad + (T_2 - T_{\Upsilon_{n-1}}) I_{\mathcal{H}^{\Upsilon_{n-1},\Upsilon_n}}(T_{\Upsilon_{n-2}}, T_{\Upsilon_{n-1}}, T_2; T_{\Upsilon_{n-1}}, T_{\Upsilon_n}) \\
&\quad + \sum_{i=1}^{n-3} (T_{\Upsilon_{i+2}} - T_{\Upsilon_{i+1}}) I_{\mathcal{H}^{\Upsilon_{i+1},\Upsilon_{i+2}}}(T_{\Upsilon_i}, T_{\Upsilon_{i+1}}, T_{\Upsilon_{i+2}}; T_{\Upsilon_{i+1}}, T_{\Upsilon_{i+2}}) \\
&\quad + \bar{\alpha}(T_{\Upsilon_1}, T_{\Upsilon_2}, T_2) I_{\mathcal{H}^{\Upsilon_1}}(t, T_{\Upsilon_1}; T_{\Upsilon_1}) \\
&\quad + \sum_{i=1}^{n-3} \bar{\alpha}(T_{\Upsilon_{i+1}}, T_2, T_{\Upsilon_{i+2}}) I_{\mathcal{H}^{\Upsilon_{i+1}}}(T_{\Upsilon_i}, T_{\Upsilon_{i+1}}; T_{\Upsilon_{i+1}}) \Big)
\end{aligned}
$$

Proof. The first three lines are easily identified as Equation 5.18, Equation 5.19 and Equation 5.22. The fourth line is the sum of Equation 5.21 as well as Equation 5.24 whereas the fifth line is the sum of Equation 5.20 and Equation 5.23. These formulae combine to become telescoping sums. We will illustrate by deducing the fifth line, starting by changing the order of the double sum:

$$
\begin{aligned}
&\sum_{i=1}^{n-3}\sum_{j=1}^{i-1} \frac{1}{\sigma}\bar{\alpha}(T_{\Upsilon_{j+1}}, T_{\Upsilon_{i+1}}, T_{\Upsilon_{i+2}}) I_{\mathcal{H}^{\Upsilon_{j+1}}}(T_{\Upsilon_j}, T_{\Upsilon_{j+1}}; T_{\Upsilon_{j+1}}) \\
&\qquad + \sum_{j=1}^{n-3} \frac{1}{\sigma}\bar{\alpha}(T_{\Upsilon_{j+1}}, T_{\Upsilon_{n-1}}, T_2) I_{\mathcal{H}^{\Upsilon_{j+1}}}(T_{\Upsilon_j}, T_{\Upsilon_{j+1}}; T_{\Upsilon_{j+1}}) \\
&\quad = \sum_{j=1}^{n-4}\sum_{i=j+1}^{n-3} \frac{1}{\sigma}\bar{\alpha}(T_{\Upsilon_{j+1}}, T_{\Upsilon_{i+1}}, T_{\Upsilon_{i+2}}) I_{\mathcal{H}^{\Upsilon_{j+1}}}(T_{\Upsilon_j}, T_{\Upsilon_{j+1}}; T_{\Upsilon_{j+1}}) \\
&\qquad + \sum_{j=1}^{n-3} \frac{1}{\sigma}\bar{\alpha}(T_{\Upsilon_{j+1}}, T_{\Upsilon_{n-1}}, T_2) I_{\mathcal{H}^{\Upsilon_{j+1}}}(T_{\Upsilon_j}, T_{\Upsilon_{j+1}}; T_{\Upsilon_{j+1}}) \\
&\quad = \sum_{j=1}^{n-4} \frac{1}{\sigma}\bar{\alpha}(T_{\Upsilon_{j+1}}, T_{\Upsilon_{j+2}}, T_2) I_{\mathcal{H}^{\Upsilon_{j+1}}}(T_{\Upsilon_j}, T_{\Upsilon_{j+1}}; T_{\Upsilon_{j+1}}) \\
&\qquad + \sum_{j=1}^{n-3} \frac{1}{\sigma}\bar{\alpha}(T_{\Upsilon_{n-2}}, T_{\Upsilon_{n-1}}, T_2) I_{\mathcal{H}^{\Upsilon_{j+1}}}(T_{\Upsilon_j}, T_{\Upsilon_{j+1}}; T_{\Upsilon_{j+1}})
\end{aligned}
$$

Now, for $j = n-4$ we get that $j+1 = n-3, j+2 = n-2$ and hence the two sums can be merged to give exactly the fifth line of the result (after renaming $j = i$). The fourth line can be treated equivalently. □

5.4.3. Stylised Examples and Discussion

We will illustrate the results of this section for two pieces of future information. This allows to draw similar graphs as in Section 5.5.3 in three dimensions. Figure 5.4.8 shows the value of the information premium for a delivery period between $T_1 = 30$

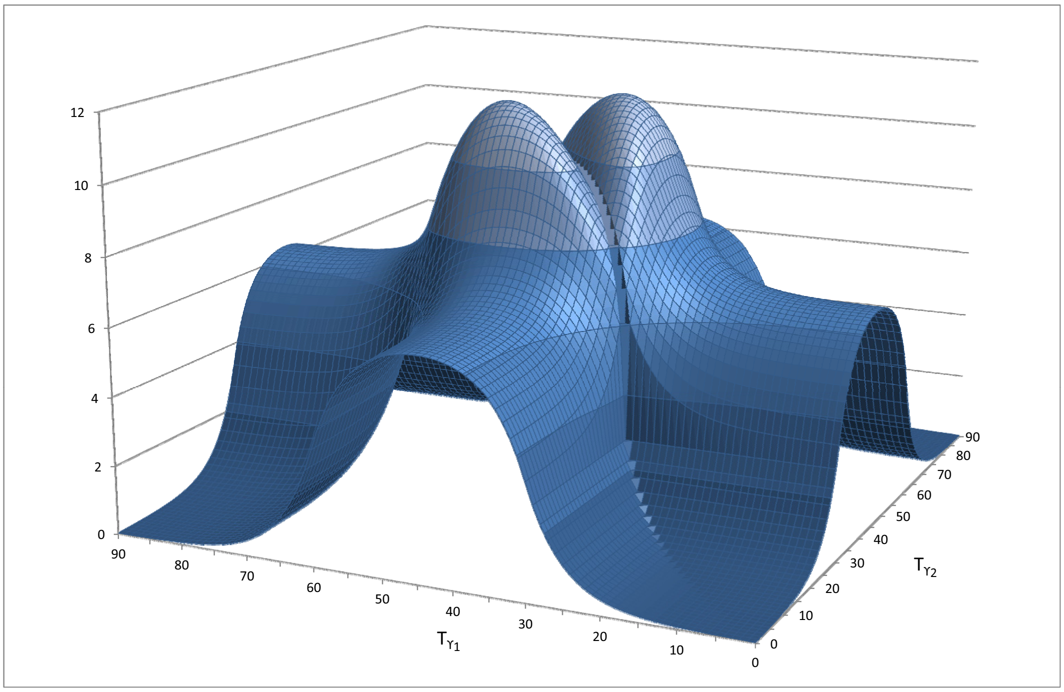


Figure 5.4.8.: Stylised information premium: Two pieces of information 1/2. In $t = 0$, delivery in $[30, 60]$, $X_0 = 0$, $\mathbb{E}[X_{T_{\Upsilon_1}} \mid \mathcal{G}_0] = 20$, $\mathbb{E}[X_{T_{\Upsilon_2}} \mid \mathcal{G}_0] = 20$ and $\alpha = 0.1$.

and $T_2 = 60$ and different combinations of T_{Υ_1} and T_{Υ_2}. Again, we have chosen $t = 0$, $X_0 = 0$ and knowledge about the future base component amounting to $\mathbb{E}[X_{T_{\Upsilon_1}} \mid \mathcal{G}_0] = \mathbb{E}[X_{T_{\Upsilon_2}} \mid \mathcal{G}_0] = 20$ €. Here, the mean-reversion parameter has been set to $\alpha = 0.1$. In case one piece of information is about time zero (i.e. no information about the future), we exhibit a shape of the information premium (in two dimensions) as before (highest value for T_{Υ_1} or T_{Υ_2} in the middle of the delivery period, zero for information long before or long after the delivery period). The highest value of the premium is taken when both pieces of information are spread out in the delivery period. If both T_{Υ_1} and T_{Υ_2} are close to each other or coincide, future information is of less value, thus the gap on the diagonal.

Two aspects of the parameter setup have been altered for Figure 5.4.9. Firstly, the speed of mean-reversion has been increased to $\alpha = 0.7$ resulting in a more angular and less curved shape of the premium (even exhibiting a plateau during the delivery period). As before, this also greatly modifies the absolute value of the premium. This exceeded more than 10 € for $\alpha = 0.1$ and is now always less than 5 €. Secondly, future information at T_{Υ_2} now amounts to 40 €, thus the loss of symmetry.

5.5. The Information Premium for Correlated Processes

In this section we will extend Benth and Meyer-Brandis (2009, Section 3.2) and show how to calculate the information premium when additional information is provided

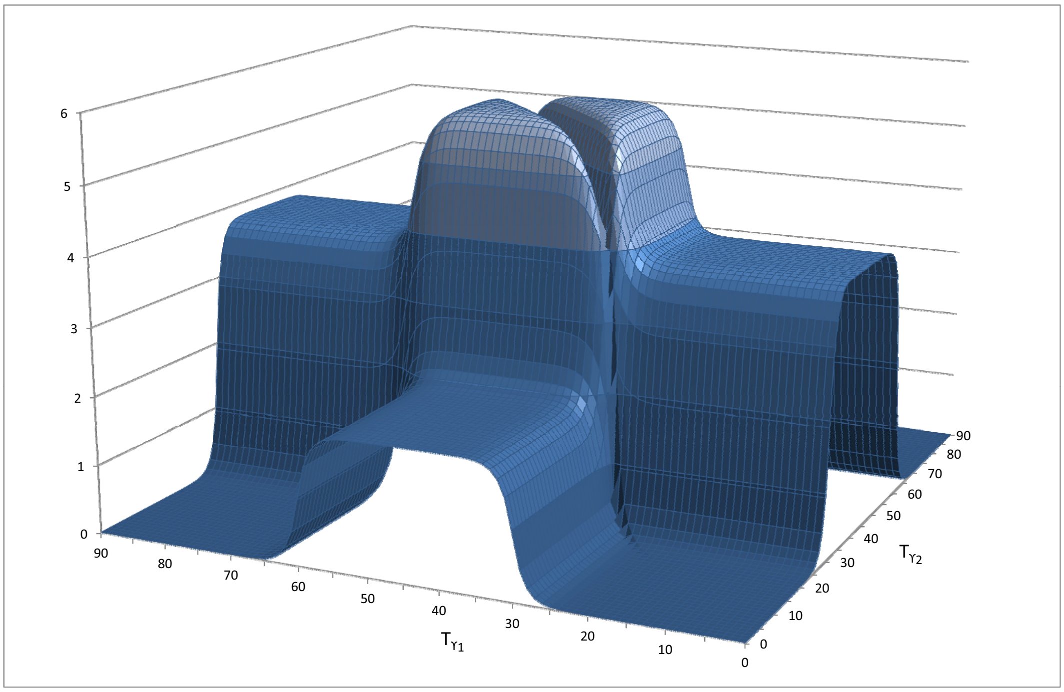


Figure 5.4.9.: Stylised information premium: Two pieces of information 2/2. In $t = 0$, delivery in $[30, 60]$, $X_0 = 0$, $\mathbb{E}[X_{T_{\Upsilon_1}} \mid \mathcal{G}_0] = 20$, $\mathbb{E}[X_{T_{\Upsilon_2}} \mid \mathcal{G}_0] = 40$ and $\alpha = 0.7$.

not about the electricity spot price directly but about a correlated process. We can consider this process as the price series of one of the fuels (gas, oil, coal), emissions or even non-financial data such as weather forecasts. In fact, we will concentrate on the temperature process as the main example in this section. The connection between the two processes will be induced by correlating the base component of the spot model with the driver of the chosen temperature model. For simplicity we will choose a Gaussian Ornstein-Uhlenbeck process with constant parameters as the stochastic part of this model. This will allow to apply the machinery discussed earlier to identify the information premium.

We remark that the sign of the resulting information premium will depend on the sign of the correlation coefficient. This will be different depending on the region we look at. For example, in Germany, heating in winter is a main price driver and we will expect higher prices with a forecast of lower temperatures (i.e. a negative correlation coefficient). The opposite can take place for regions with a high degree of air conditioning such as California or the South of the US.

5.5.1. The Temperature Model

The simple mean-reverting process is one of the processes used in the literature to model temperature. Examples are Benth and Ŝaltyte-Benth (2005), Benth and Ŝaltyte-Benth (2007) or Dornier and Querel (2000) (the latter using data from Chicago airport). As a more sophisticated model, it has been shown in Benth et al.

(2007b) that an auto-regressive process with lag three is already performing very well for the temperature in Stockholm, Sweden. Furthermore, the authors propose improving their model by adding a seasonal speed of mean reversion. This model was also applied to the temperature in Berlin in López Cabrera and Härdle (2011). The authors compare implied and observed prices of weather derivatives and deduce a non-zero market price of risk.

Here, as mentioned before, we will consider constant parameters and an $AR(1)$ (Ornstein-Uhlenbeck) process as to allow for closed-form solutions. We remark that this will provide the toolbox to deal with more sophisticated (Gaussian) models.

Definition 5.5.1. The temperature process and correlation. We will denote by Z_t the temperature (in degrees). Z_t will satisfy the stochastic differential equation:

$$dZ_t = -\alpha_Z(Z_t - \mu_Z(t))dt + \sigma_Z dW_t^Z \tag{5.25}$$

where α_Z, σ_Z are constant and $\mu_Z(t)$ is a time-dependent deterministic function modelling seasonalities. Furthermore, we will interconnect temperature with electricity by introducing a correlation coefficient ρ in the following manner

$$dW_t dW_t^Z = \rho dt \tag{5.26}$$

where W_t is the driving Brownian motion of the base component of the spot price (cf. Equation 3.2).

By standard theory we can then write the base component Brownian motion of the spot as

$$dW_t = \rho dW_t^Z + \sqrt{1-\rho^2} dW_t^S \tag{5.27}$$

where W^S is another Brownian motion capturing spot specific deviations.

The relevant historical filtration is now given by

$$\mathcal{F}_t = \sigma(W_s^S; 0 \le s \le t) \vee \sigma(W_s^Z; 0 \le s \le t) \vee \sigma(L_s; 0 \le s \le t) \tag{5.28}$$

including temperature information.

5.5.2. Calculating the Information Premium

We want to enlarge the historical filtration by a weather forecast, i.e. by some future value of the temperature process Z_{T_Υ}. As before, we are going to use *Imkeller's method* (see Theorem 2.2.8) to find the martingale decomposition of the temperature Brownian motion W_t^Z under the enlarged filtration.

Filtration $\mathcal{H}_t$ is defined by

$$\mathcal{H}_t = \mathcal{F}_t \vee \sigma\left(Z_{T_\Upsilon}\right) = \mathcal{F}_t \vee \sigma\left(\int_t^{T_\Upsilon} e^{-\alpha_Z(T_\Upsilon - s)} dW_s^Z\right) \tag{5.29}$$

Thus, reproducing the calculations from Section 5.2.3 we can state the martingale decomposition of W_t^Z under the enlarged filtration $\mathcal{H}_t$:

$$\xi_t^Z = W_t^Z - \int_0^t \frac{2\alpha_Z e^{\alpha_Z s}}{e^{2\alpha_Z T_\Upsilon} - e^{2\alpha_Z s}} \left(\int_s^{T_\Upsilon} e^{\alpha_Z u} dW_u^Z\right) ds \tag{5.30}$$

where as usual ξ_t^Z is an $\mathcal{H}_t$-martingale. Remembering Equation 5.10 we will denote by $a^Z(t)$ a version of $a(t)$ with temperature parameters.

Now, we can commence calculating the information premium.

Proposition 5.5.1. Information premium with weather forecasts. *The information premium under the filtration $\mathcal{G}_t \subseteq \mathcal{F}_t \vee \sigma(Z_{T_\Upsilon})$ and for $0 \leq t \leq T_1 < T_2 \leq T_\Upsilon$ is given by*

$$
\begin{aligned}
&I_{\mathcal{G}}(t, T_1, T_2; Z, T_\Upsilon) \\
&= \frac{\rho A(t, T_1, T_2; T_\Upsilon)}{T_2 - T_1} \left(\mathbb{E}[Z_{T_\Upsilon} \mid \mathcal{G}_t] - e^{-\alpha_Z (T_\Upsilon - t)} Z_t - \int_t^{T_\Upsilon} \alpha_Z \mu_Z(s) e^{-\alpha_Z (T_\Upsilon - s)} ds \right)
\end{aligned}
$$

The deterministic function $A(t, T_1, T_2; T_\Upsilon)$ satisfies

$$
A(\cdot) = \frac{2\alpha_Z \sigma e^{\alpha_Z (T_\Upsilon + t)} \left(\frac{1}{\alpha_Z} \left(e^{\alpha_Z (T_2 - t)} - e^{\alpha_Z (T_1 - t)} \right) + \frac{1}{\alpha} \left(e^{-\alpha (T_2 - t)} - e^{-\alpha (T_1 - t)} \right) \right)}{\sigma_Z (\alpha + \alpha_Z)(e^{2\alpha_Z T_\Upsilon} - e^{2\alpha_Z t})}
$$

Proof. The definition of the information premium is

$$
I_{\mathcal{G}}(t, T_1, T_2; Z, T_\Upsilon) = \frac{1}{T_2 - T_1} \left(\mathbb{E}\left[\int_{T_1}^{T_2} S_u du \mid \mathcal{G}_t \right] - \mathbb{E}\left[\int_{T_1}^{T_2} S_u du \mid \mathcal{F}_t \right] \right)
$$

As usual, terms in t cancel as both filtrations coincide. Thus,

$$
\begin{aligned}
&I_{\mathcal{G}}(t, T_1, T_2; Z, T_\Upsilon) \\
&= \frac{1}{T_2 - T_1} \mathbb{E}\left[\int_{T_1}^{T_2} \left(\sigma\rho \int_t^u e^{-\alpha(u-s)} dW_s^Z + \sigma\sqrt{1-\rho^2} \int_t^u e^{-\alpha(u-s)} dW_s^S \right) du \mid \mathcal{G}_t \right] \\
&= \frac{1}{T_2 - T_1} \mathbb{E}\left[\int_{T_1}^{T_2} \sigma\rho \int_t^u e^{-\alpha(u-s)} dW_s^Z du \mid \mathcal{G}_t \right]
\end{aligned}
$$

Under $\mathcal{G}_t$, the spot-intrinsic process W^S is still a martingale and thus the expectation of the second integral is zero. Then, we substitute the $\mathcal{G}_t$-decomposition of W^Z

$$
\begin{aligned}
&I_{\mathcal{G}}(t, T_1, T_2; Z, T_\Upsilon) \\
&= \frac{1}{T_2 - T_1} \mathbb{E}\left[\int_{T_1}^{T_2} \sigma\rho \int_t^u e^{-\alpha(u-s)} d\left(\xi_s^Z + \int_0^s a^Z(v) \int_v^{T_\Upsilon} e^{\alpha_Z w} dW_w^Z dv \right) du \mid \mathcal{G}_t \right] \\
&= \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \sigma\rho \int_t^u e^{-\alpha(u-s)} a^Z(s) \mathbb{E}\left[\int_s^{T_\Upsilon} e^{\alpha_Z v} dW_v^Z \mid \mathcal{G}_t \right] ds\, du \\
&= \frac{1}{T_2 - T_1} \mathbb{E}\left[\int_t^{T_\Upsilon} e^{\alpha_Z v} dW_v^Z \mid \mathcal{G}_t \right] \int_{T_1}^{T_2} \sigma\rho \int_t^u \frac{e^{-\alpha(u-s)} 2\alpha_Z e^{\alpha_Z s}}{e^{2\alpha_Z T_\Upsilon} - e^{2\alpha_Z t}} ds\, du
\end{aligned}
$$

Here, we have applied Lemma 5.3.4 once more. Now, solving the remaining double-integral and rewriting in terms of Z rather than W^Z provides the required result, in particular function $A(t, T_1, T_2; T_\Upsilon)$. □

A first interpretation of this result shows that the sign of the information premium depends on two things: firstly, and as before, the relationship between the magnitude of the weather forecast and the current temperature. Secondly, a change in sign of the correlation coefficient ρ will result in a change in sign of the premium. The situation of Germany or Scandinavia (lower temperature results in higher electricity prices) would feature a $\rho < 0$ whereas the situation of California would be given by $\rho > 0$.

We remark that, as in Section 5.3.2, it is possible to recover the delivery time result from Benth and Meyer-Brandis (2009, Proposition 3.5) by letting $T_2 \to T_1$:

Corollary 5.5.1. Limit of weather information premium. *The following holds:*

$$\lim_{T_2 \to T_1} I_{\mathcal{G}}(t, T_1, T_2; Z, T_\Upsilon) = I_{\mathcal{G}}(t, T_1; Z, T_\Upsilon)$$

where

$$I_{\mathcal{G}}(t, T_1; Z, T_\Upsilon) = \rho \, \frac{2\alpha_Z \sigma e^{\alpha_Z (T_\Upsilon + T_1)} \left(1 - e^{-(\alpha + \alpha_Z)(T_1 - t)}\right)}{\sigma_Z (\alpha + \alpha_Z)(e^{2\alpha_Z T_\Upsilon} - e^{2\alpha_Z t})} \left(\mathbb{E}[Z_{T_\Upsilon} \mid \mathcal{G}_t] - e^{-\alpha_Z (T_\Upsilon - t)} Z_t - \int_t^{T_\Upsilon} \alpha_Z \mu_Z(s) e^{-\alpha_Z (T_\Upsilon - s)} ds \right)$$

For other orderings of time points we have the usual versions of the information premium which we will state omitting proofs.

Lemma 5.5.1. Information premium (temperature information before delivery). *For $0 \le t < T_\Upsilon \le T_1 < T_2$ the information premium is given by*

$$I_{\mathcal{G}}(t, T_1, T_2; Z, T_\Upsilon) = \frac{1}{T_2 - T_1} \bar{\alpha}(T_\Upsilon, T_1, T_2) I_{\mathcal{G}}(t, T_\Upsilon; Z, T_\Upsilon)$$

where the delivery time information premium is as in Corollary 5.5.1.

Again, when additional information is provided during the delivery period, we find the information premium to be a linear combination of Proposition 5.5.1 and Lemma 5.5.1.

Lemma 5.5.2. Information premium (temperature information during delivery). *For $0 \le t \le T_1 < T_\Upsilon < T_2$ the information premium is given by*

$$I_{\mathcal{G}}(t, T_1, T_2; Z, T_\Upsilon) = \frac{T_\Upsilon - T_1}{T_2 - T_1} \underbrace{I_{\mathcal{G}}(t, T_1, T_\Upsilon; Z, T_\Upsilon)}_{\textit{Proposition } 5.5.1} + \frac{T_2 - T_\Upsilon}{T_2 - T_1} \underbrace{I_{\mathcal{G}}(t, T_\Upsilon, T_2; Z, T_\Upsilon)}_{\textit{Lemma } 5.5.1}$$

Summarising, we have seen that by making use of a correlation coefficient and the usual machinery we can calculate the information premium for given information about a related asset or, more generally, any process related to the spot price.

5.5.3. Stylised Examples and Discussion

We will conclude this section with another set of graphs to illustrate the information premium with temperature forecasts. The setup will be as follows: We will assume that the current ($t = 0$) temperature and its mean level are ten degrees, i.e. $Z_0 = \mu_Z(t) = 10$. Furthermore, the base component in $t = 0$ will take value $X_0 = 0$. As before in this chapter, we will consider a delivery period in $[30, 60]$. The volatility of electricity will be $\sigma = 5$ whereas $\sigma^Z = 1$. We will use a negative correlation of $\rho = -0.5$ to mimic a Central European situation.

Figure 5.5.10 depicts the graph of the information premium for various values of T_Υ as well as temperature forecasts between -15 and $+15$ degrees. We find a negative value for the premium for forecasts that are greater than ten degrees, i.e. warmer temperature than expected by μ^Z around the time. If it is colder than ten degrees at T_Υ, more heating is needed and the premium becomes positive. As before, relatively large rates of mean-reversion imply the angular shape.

Two details of the setup are modified for Figure 5.5.11. Firstly, the correlation is now $\rho = 0.5$ so that we are in a California type situation. Clearly, the graph now features a lesser premium for lesser forecasts as can be expected for regions with cooling. Secondly, the mean-reversion of electricity has been increased whereas that of the temperature process has been decreased ($\alpha = 0.5, \alpha^Z = 0.1$), i.e. to more realistic values. We now experience a non-symmetrical premium which takes its maximum more towards the end of the delivery period. This maximum is of slightly less absolute value than before.

5.6. Contribution and Discussion

In this chapter we find formulae for the information premium for different types of future information, different arrangements of the time axis, using different methods and, most importantly, for forward contracts with a delivery period. This type of contract is the one traded on electricity exchanges and thus of the greatest relevance, especially when it comes to empirical investigations.

The first part of this chapter gives an overview of the methods available to identify the information yield depending on the structure of the additional information available under the market filtration. We find that Imkeller's method involving Malliavin's calculus greatly decreases the length of the necessary calculations.

Then, we adapt the definition of the information premium to take delivery periods into consideration. Taking the relevant case of non-precise information about the base component of the spot model as an example we then derive formulae for different arrangements of the time axis, depending on whether the future information is given during, before or after the delivery period. We redevelop the formulae from Benth and Meyer-Brandis (2009) by taking limits of our more general results.

The results for one piece of future information are then generalised first to two pieces of future information and these in turn are used to come up with the general information premium with delivery period and any number of pieces of future information.

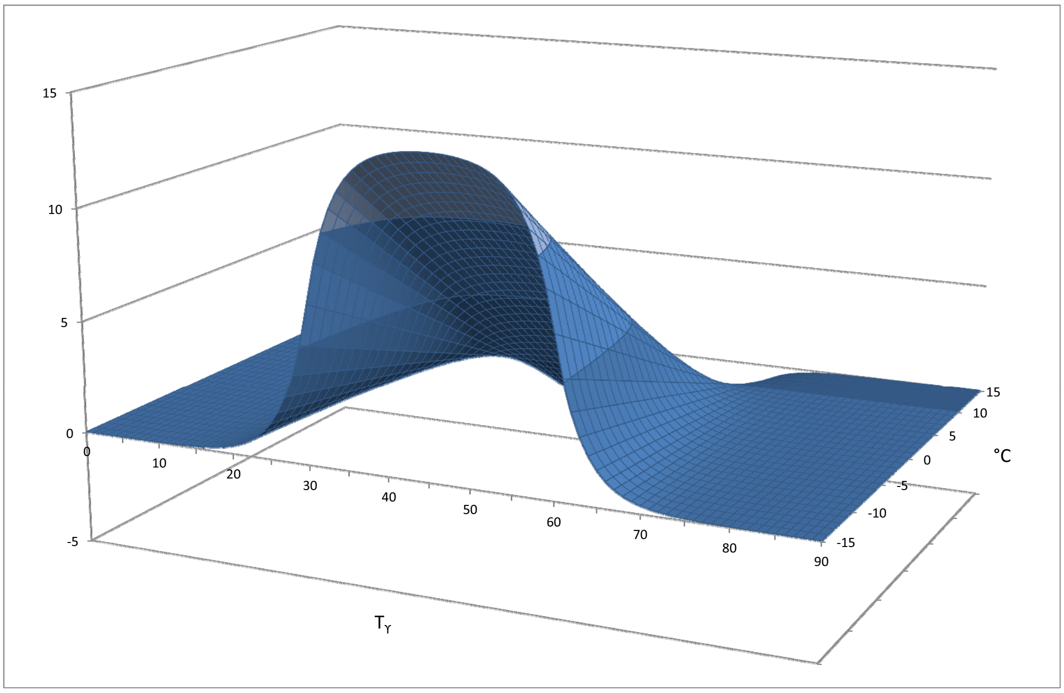


Figure 5.5.10.: Stylised information premium: Corr. temperature 1/2. Different forecasts and T_Υ, $t = 0$, $[30, 60]$, $X_0 = 0$, $Z_0 = \mu^Z(t) = 10$, $\alpha = \alpha^Z = 0.3$, $\rho = -0.5$.

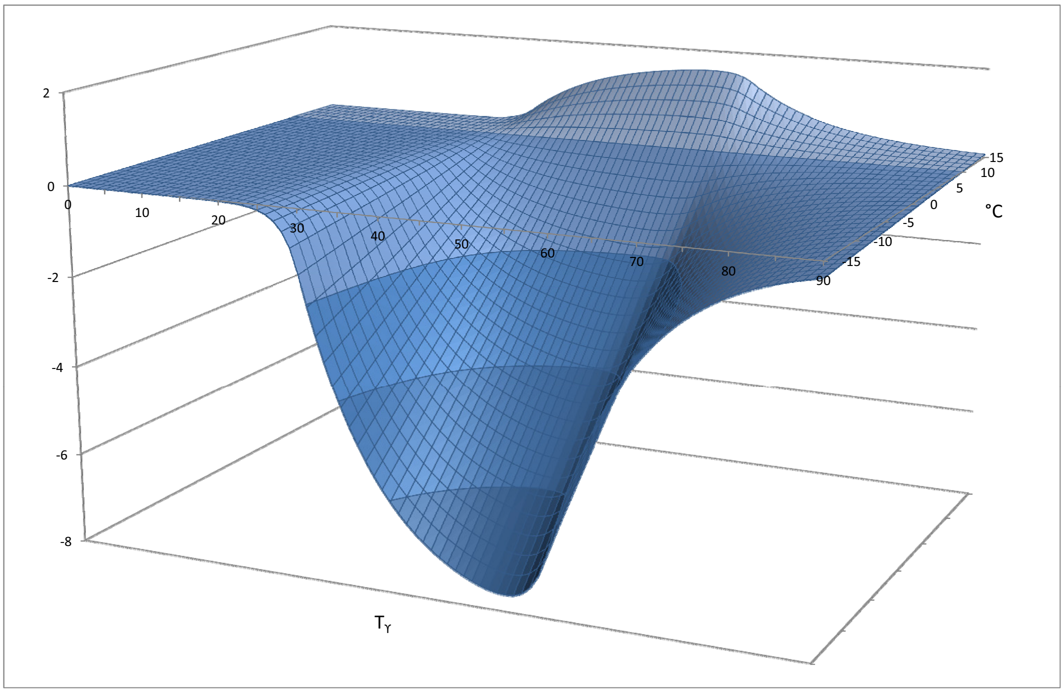


Figure 5.5.11.: Stylised information premium: Corr. temperature 2/2. Different forecasts and T_Υ, $t = 0$, $X_0 = 0$, $Z_0 = \mu^Z(t) = 10$, $\alpha = 0.5$, $\alpha^Z = 0.1$, $\rho = 0.5$.

This basically leaves one more aspect to be considered, namely the case that market traders possess information not about the future electricity spot directly but about some related relevant process. Taking temperature forecasts and a Gaussian correlation model as an example we also derive closed-form solutions for all cases of this framework.

Each of the sections of this chapter also features stylised examples and illustrative figures that deliver an insight into shape and size of the information premium.

Chapter 6.

Option Pricing and Information Approach

6.1. Literature Overview and Summary

After having identified the information premium for various scenarios, we will, in this chapter, present and discuss the problems that arise when pricing options on forwards in the presence of additional information. We remark that the basis of this chapter is the paper Benth, Biegler-König, and Kiesel (2013b).

In Section 6.4, we will calculate prices in closed-form of vanilla options under both the historical filtration as well as under the (enlarged) market filtration. To this end, once again, we will only consider the Gaussian component of our spot model.

Before we begin calculations, though, we will try to relate our ideas and the information approach to existing ideas from the literature. In particular, in the context of financial modelling, the mathematical technique of enlargement of filtrations has been applied intensively in modelling insider trading. The corresponding publications normally assume that there exists an insider who has access to (additional) future information. Thus, the equivalent of what we have called the market filtration in this thesis would be the so-called insider filtration. Examples of this branch of the literature are, amongst others, Pikovsky and Karatzas (1996), Elliott et al. (1997), Hu and Øksendal (2007), Amendinger et al. (2003), Amendinger et al. (1998), Biagini and Øksendal (2005) and Rindisbacher (2010), who all consider enlargement of filtrations in a Brownian framework and with a logarithmic utility function. Campi (2005) addresses the question of how both types of traders set up hedging portfolios. More recently, insider trading on Lévy-driven markets has been the subject of papers like Nunno et al. (2005), Ankirchner (2008), Kohatsu-Higa and Yamazato (2008) or Ankirchner and Zwierz (2011). Furthermore, parts of the two dissertations Ankirchner (2005) and Amendinger (1999) discuss various aspects of such an insider trading framework.

Generally, all of the examples mentioned above concentrate on comparing the utility of the two types of investors. The reasoning behind this utility approach is that it turns out that both investors assign the same monetary value to most derivatives. We will discuss the reasons in detail in Section 6.2 also providing some general results. Trivially speaking, we have seen in Section 2.2 that enlarging the filtration changes drift terms while the volatility remains unchanged. And drifts disappear under the pricing measure. Still, we will see that for our special underlying electricity this is not the end of the story and we will discuss and prepare the calculations to follow in Section 6.3. In particular, the question of what exactly will be an asset in the traditional sense on electricity markets will turn out to be of the utmost importance.

6.2. Insider Trading and Option Pricing

We will now show that derivatives written on traditional underlyings have the same value under an enlarged filtration and under the historical filtration. We will follow the ideas and the framework laid out in Amendinger (1999) and previously discussed in Section 2.2. We know from standard financial mathematics that a market is free of arbitrage whenever there exists a risk-neutral measure equivalent to the real-world measure such that discounted assets are martingales. Furthermore, when considering complete financial markets we know that this measure is unique (cf. Shreve (2004, Section 5.3.2) or Bingham and Kiesel (2004, Section 4.4) for fundamental theorems). Under certain mild assumptions, Theorem 2.2.6 states that we can find a special, unique (risk-neutral) measure such that a martingale under the historical filtration is also a martingale under the enlarged filtration. This tells us that if the no-arbitrage and completeness properties are satisfied for the ordinary investor, this is also the case for the insider (there is a one-to-one relationship between the risk-neutral measures under the historical and the enlarged filtrations).

On such complete and arbitrage-free markets we have available a theoretical way to perfectly replicate a contingent claim with known payoff at maturity. The well-known Delta-hedge for vanilla options is one example. The way of setting up a replicating portfolio is clearly not only available to the normal (or honest) trader but also to the insider. Thus, intuitively, as both traders can construct a perfect replication of the option, the prices they assign will coincide, too.

In more technical terms, we recall that the payoff of an option with maturity at some time T can be represented by an $\mathcal{F}_T$-measurable random variable H. The fair value of this claim in $t < T$ is then given by the risk-neutral valuation formula, i.e. it is the discounted risk-neutral expectation of H. Let $\mathbb{Q}_\mathcal{F}$ and $\mathbb{Q}_\mathcal{G}$ denote the risk-neutral measures on $(\Omega, \mathcal{F})$ and $(\Omega, \mathcal{G})$, respectively. As in Section 2.2 we define $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(G)$ for some random variable G taking values in $(U,\mathcal{U})$. The following proposition, taken from Amendinger (1999, Theorem 4.6), will prove that both risk-neutral formulae result in the same value. We will provide a more detailed proof.

Proposition 6.2.1. Equivalence of option prices. *With the stricter assumption described in Lemma 2.2.1 we have for an $\mathcal{F}_T$-measurable random variable H with $\mathbb{E}^{\mathbb{Q}_\mathcal{F}}[H] < \infty$ that*

$$\mathbb{E}^{\mathbb{Q}_\mathcal{F}}[H|\mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}_\mathcal{G}}[H|\mathcal{G}_t]$$

for all $0 \le t \le T$.

Proof. Remembering the notation of Theorem 2.2.6 we let Z_t be the Radon-Nikodým derivative of $\mathbb{Q}_\mathcal{F}$. Also, we let p_t^G be the conditional density process of G. Then, as in Lemma 2.2.1, we define measure $\mathbb{Q}_\mathcal{G}$ by the density $\frac{Z_T}{p_T^G}$. We prove that the expectation of both conditional expectations is equal for arbitrary $A_t \in \mathcal{F}_t$ and $B \in \mathcal{U}$.

$$\mathbb{E}^{\mathbb{Q}_\mathcal{G}}\left[\mathbb{E}^{\mathbb{Q}_\mathcal{G}}[H \mid \mathcal{G}_t]\mathbb{1}_{A_t \cap \{G \in B\}}\right] = \mathbb{E}^{\mathbb{Q}_\mathcal{G}}\left[H \mathbb{1}_{A_t \cap \{G \in B\}}\right]$$

because of the tower property. Then:

$$\mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}\left[H\mathbb{1}_{A_t\cap\{G\in B\}}\right] = \mathbb{E}\left[\frac{Z_T}{p_T^G}H\mathbb{1}_{A_t}\mathbb{1}_{\{G\in B\}}\right] = \mathbb{E}\left[Z_T H\mathbb{1}_{A_t}\mathbb{E}\left[\frac{1}{p_T^G}\mathbb{1}_{G\in B}\mid \mathcal{F}_T\right]\right]$$

where in the last step we have used the tower property again. We continue by transforming the inner expectation:

$$\mathbb{E}\left[\frac{1}{p_T^G}\mathbb{1}_{G\in B}\mid \mathcal{F}_T\right] = \int_B \frac{1}{p_T^G}p_T^G P(G\in dl) = P(G\in B)$$

and this is a constant probability and thus practically the decoupling property of part three of Lemma 2.2.1. We consider the whole expression again, making use of the tower property once more:

$$\mathbb{E}\left[Z_T H\mathbb{1}_{A_t}\right]P(G\in B) = \mathbb{E}^{\mathbb{Q}_{\mathcal{F}}}\left[H\mathbb{1}_{A_t}\right]P(G\in B) = \mathbb{E}^{\mathbb{Q}_{\mathcal{F}}}\left[\mathbb{E}[H\mid\mathcal{F}_t]\mathbb{1}_{A_t}\right]P(G\in B)$$

and now we change the outer measure again:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}_{\mathcal{F}}}\left[\mathbb{E}[H\mid\mathcal{F}_t]\mathbb{1}_{A_t}\right]P(G\in B) &= \mathbb{E}\left[Z_T\frac{Z_t}{Z_t}\mathbb{E}[H\mid\mathcal{F}_t]\mathbb{1}_{A_t}\right]P(G\in B)\\
&= \mathbb{E}\left[Z_t\mathbb{E}^{\mathbb{Q}_{\mathcal{F}}}[H\mid\mathcal{F}_t]\mathbb{1}_{A_t}\frac{1}{p_t^G}\mathbb{1}_{G\in B}\right]\\
&= \mathbb{E}\left[\mathbb{E}\left[\frac{Z_T}{p_T^G}\mid\mathcal{F}_t\right]\mathbb{E}^{\mathbb{Q}_{\mathcal{F}}}[H\mid\mathcal{F}_t]\mathbb{1}_{A_t\cap\{G\in B\}}\right]\\
&= \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}\left[\mathbb{E}^{\mathbb{Q}_{\mathcal{F}}}[H\mid\mathcal{F}_t]\mathbb{1}_{A_t\cap\{G\in B\}}\right]
\end{aligned}$$

here, we have used the martingale property of $\frac{Z}{p^G}$ (i.e. the first statement of Lemma 2.2.1) and the tower property. This establishes the result. □

Summarising, we have proved that, at least for a traditional underlying, prices of (vanilla) options are the same for both the insider and the honest (ordinary) investor.

6.3. Assets, Risk-neutrality

We are facing a different situation when the underlying is electricity. The spot is non-storable and thus not an asset in the classical sense as it is not tradable. This poses a number of questions when trying to price forwards and options on forwards. For example, one might ask whether the results from the literature can be translated, i.e. that options have identical prices under both filtrations. In the end, as the spot is not tradable, one cannot follow the traditional argument and compare hedges. However, forwards are traded assets and we now have two versions of the forward price: one under the historical and one under the market filtration. Hence, it is difficult to assign to each filtration one type of investor (as in the insider literature) and to consider both investors coexisting on the market. For the underlying electricity we are facing an information asymmetry and we should think of two different models, i.e. $(S_t, \mathcal{F}_t)$ for the spot price and $(S_t, \mathcal{G}_t)$ for the forward and derivative markets.

Summarising, our way to interpret the objects discussed previously is as follows: the informed and the uninformed traders calculate two sets of prices for themselves depending on their best knowledge. Our analysis consequently ignores the question of how observed market forward prices are then amalgamated from these two individual sets of prices.

6.4. Prices of Vanilla Options on Forward Contracts

After having discussed the literature and the relevant framework we now want to price a plain vanilla call on a forward on electricity. We will assume that the option expires in T and the underlying forward has, as usual, delivery period between T_1 and T_2. Furthermore, there is relevant additional future information in T_Υ. This setup is illustrated in Figure 6.4.1. Remembering the formulae from Section 5.3 we remark that T_Υ could, of course, be any time after T. The following discussion would not have to be fundamentally modified.

Figure 6.4.1.: Option prices: Setup of the time axis. For an option with maturity T on a forward with delivery in $[T_1, T_2]$ and additional information at time T_Υ.

As mentioned in Section 6.1, we will assume that the spot follows a standard Gaussian Ornstein-Uhlenbeck process with constant parameters, i.e. the base component of the model of Equation 3.2, only:

$$X_T = e^{-\alpha(T-t)} X_t + \sigma \int_t^T e^{-\alpha(T-u)} dW_u$$

There are two reasons for this: on the one hand, closed-form solutions will only be available for a Gaussian model. For Lévy models, standard numerical methods such as the fast Fourier analysis (discussed for example in Benth et al. (2008b, Section 9.1.2) and presented in Carr and Madan (1999)) can be applied without much further work and as an extension. On the other hand, we want to put emphasis on the methodology, thus we leave out the seasonality for notational clarity. We will later, in Section 6.5, add a constant seasonality that will yield more realistic figures and results.

The forward price with delivery period and under the real-world measure was calculated in Proposition 3.4.2 and takes the form

$$F_{\mathcal{F}}^{\mathbb{P}}(t, T_1, T_2) = \frac{1}{T_2 - T_1} \, \bar{\alpha}(t, T_1, T_2) X_t$$

where, as usual, the auxiliary function $\bar{\alpha}(t, T_1, T_2)$ is defined as in Equation 5.10.

Now, we calculate the forward dynamics:

$$dF^{\mathbb{P}}_{\mathcal{F}}(t,T_1,T_2) = \frac{1}{T_2-T_1}\ \left(d\bar{\alpha}(t,T_1,T_2)X_t + \bar{\alpha}(t,T_1,T_2)dX_t\right)$$

The function $\bar{\alpha}(t,T_1,T_2)$ is deterministic and we have $t < T_1$. Thus

$$d\bar{\alpha}(t,T_1,T_2) = d\left(-\frac{1}{\alpha}\left(e^{-\alpha(T_2-t)} - e^{-\alpha(T_1-t)}\right)\right) = \alpha\bar{\alpha}(t,T_1,T_2)dt$$

Hence, substituting the dynamics of the Ornstein-Uhlenbeck process

$$\begin{aligned} dF^{\mathbb{P}}_{\mathcal{F}}(t,T_1,T_2) &= \frac{1}{T_2-T_1}\ \left(\bar{\alpha}(t,T_1,T_2)(-\alpha X_t dt + \sigma dW_t) + \alpha\bar{\alpha}(t,T_1,T_2)X_t dt\right) \\ &= \frac{1}{T_2-T_1}\ \sigma\bar{\alpha}(t,T_1,T_2)dW_t \end{aligned} \tag{6.1}$$

Now, terms in dt are zero and the $(\mathcal{F},\mathbb{P})$-Brownian motion W_t is already a martingale. Integrating yields:

$$F^{\mathbb{P}}_{\mathcal{F}}(T,T_1,T_2) = F^{\mathbb{P}}_{\mathcal{F}}(t,T_1,T_2) + \frac{1}{T_2-T_1}\ \sigma\int_t^T \bar{\alpha}(s,T_1,T_2)dW_s \tag{6.2}$$

The electricity market is incomplete and we can choose our risk-neutral pricing measure. Equation 6.2 suggests, using notation from Proposition 6.2.1, that $\mathbb{Q}_{\mathcal{F}} = \mathbb{P}$.

Starting with Equation 6.1, we rewrite the forward dynamics for the enlarged market filtration $\mathcal{G}_t$ and in terms of the information drift $\mu^{\mathcal{G}}_t$ as given by Definition 4.3.1:

$$\begin{aligned} dF^{\mathbb{P}}_{\mathcal{G}}(t,T_1,T_2) &= \frac{1}{T_2-T_1}\ \sigma\bar{\alpha}(t,T_1,T_2)\ d\left(\xi_t + \int_0^t \mu^{\mathcal{G}}_s ds\right) \\ &= \frac{1}{T_2-T_1}\ \left(\sigma\bar{\alpha}(t,T_1,T_2)d\xi_t + \sigma\bar{\alpha}(t,T_1,T_2)\mu^{\mathcal{G}}_t dt\right) \end{aligned} \tag{6.3}$$

Again, we integrate the dynamics:

$$\begin{aligned} &F^{\mathbb{P}}_{\mathcal{G}}(T,T_1,T_2) \\ &\quad = F^{\mathbb{P}}_{\mathcal{G}}(t,T_1,T_2) + \frac{\sigma}{T_2-T_1}\left(\int_t^T \bar{\alpha}(s,T_1,T_2)d\xi_t + \int_t^T \bar{\alpha}(s,T_1,T_2)\mu^{\mathcal{G}}_s ds\right) \end{aligned} \tag{6.4}$$

Section 2.2 tells us that this is a $(\mathcal{G},\mathbb{P})$-semimartingale. In particular, the dt terms are $\mathcal{G}_s$-measurable; thus we can change measure to obtain martingale dynamics under $\mathcal{G}$ and the new measure $\mathbb{Q}_{\mathcal{G}}$. This connection was discovered in Protter (1989) and Föllmer and Imkeller (1993). We follow the notation and approach of the former and define new processes

$$\begin{aligned} M_t &= \int_0^t \left(-\mu^{\mathcal{G}}_s\right) d\xi_s \\ N_t &= 1 + \int_0^t N_s dM_s \end{aligned}$$

Clearly, N_t is an exponential martingale and has a well-known solution for $s < t$

$$N_t = N_s \exp\left(-\int_s^t \frac{1}{2}\left(\mu_u^{\mathcal{G}}\right)^2 du - \int_s^t \mu_u^{\mathcal{G}} d\xi_u\right)$$

As N_t has expectation one (i.e. $N_t \in L^1(\mathcal{G}, \mathbb{P})$), we can now apply the Girsanov-Meyer theorem (see Dellacherie and Meyer (1982, page 238) or Protter (2005, page 135) for more details) with derivatives

$$\left.\frac{d\mathbb{Q}_{\mathcal{G}}}{d\mathbb{P}}\right|_{\mathcal{G}_t} = N_t \ , \ \left.\frac{d\mathbb{P}}{d\mathbb{Q}_{\mathcal{G}}}\right|_{\mathcal{G}_t} = N_t^{-1}$$

The theorem states that the $\mathbb{Q}_{\mathcal{G}}$-decomposition of the $(\mathcal{G}, \mathbb{P})$-Brownian motion ξ_t is

$$\xi_t = \left(\xi_t - \int_0^t \frac{1}{N_s} \ d < N, \xi >_s\right) + \int_0^t \frac{1}{N_s} \ d < N, \xi >_s$$

Calculating the integral yields

$$\begin{aligned}
\int_0^t \frac{1}{N_s} \ d < N, \xi >_s &= \int_0^t \frac{1}{N_s} \ d < \int_0^{\cdot} (-N_u \mu_u^{\mathcal{G}}) d\xi_u, \int_0^{\cdot} d\xi_u >_s \\
&= \int_0^t \frac{1}{N_s} \ d\left(\int_0^s (-N_u \mu_u^{\mathcal{G}}) du\right) \\
&= \int_0^t \frac{1}{N_s} (-N_s \mu_s^{\mathcal{G}}) ds \\
&= \int_0^t -\mu_s^{\mathcal{G}} ds
\end{aligned}$$

so that under $(\mathcal{G}, \mathbb{Q}_{\mathcal{G}})$ we have the decomposition

$$\xi_t = \left(\xi_t + \int_0^t \mu_s^{\mathcal{G}} ds\right) - \int_0^t \mu_s^{\mathcal{G}} ds = W_t - \int_0^t \mu_s^{\mathcal{G}} ds \tag{6.5}$$

This means that the original $(\mathcal{F}, \mathbb{P})$-Brownian motion W_t is also a Brownian motion under $(\mathcal{G}, \mathbb{Q}_{\mathcal{G}})$. We remark that this is, of course, Theorem 2.2.6 in disguise. Consequently, rewriting Equation 6.3, the forward dynamics under $(\mathcal{G}, \mathbb{Q}_{\mathcal{G}})$ are

$$\begin{aligned}
dF_{\mathcal{G}}^{\mathbb{Q}_{\mathcal{G}}}(t, T_1, T_2) &= \frac{\sigma}{T_2 - T_1} \bar{\alpha}(t, T_1, T_2) d\left(W_t - \int_0^t \mu_s^{\mathcal{G}} ds\right) + \frac{\sigma}{T_2 - T_1} \bar{\alpha}(t, T_1, T_2) \mu_t^{\mathcal{G}} dt \\
&= \frac{1}{T_2 - T_1} \sigma \bar{\alpha}(t, T_1, T_2) dW_t^{\mathbb{Q}_{\mathcal{G}}} \qquad (6.6)
\end{aligned}$$

Hence, the forward price is a martingale. Integrating,

$$F_{\mathcal{G}}^{\mathbb{Q}_{\mathcal{G}}}(T, T_1, T_2) = F_{\mathcal{G}}^{\mathbb{Q}_{\mathcal{G}}}(t, T_1, T_2) + \frac{1}{T_2 - T_1} \sigma \int_t^T \bar{\alpha}(t, T_1, T_2) dW_s^{\mathbb{Q}_{\mathcal{G}}} \tag{6.7}$$

Comparing Equation 6.7 and Equation 6.2 we find that only prices at time t are different whereas the other terms are the same. We now have the ingredients to calculate options on futures under the two filtrations. The crucial difference between the insider literature and our analysis is that, although we replicate the result that the underlying has the same dynamics under both filtrations, we have different starting values in t because the additional information induces a different model for forward pricing.

6.4.1. The Traditional Price of a Vanilla Call

The standard case in the literature is that of pricing an option on a forward, both under the historical filtration, i.e. our forward model is now $(\mathcal{F}_t, \mathbb{P})$. In order to do so we need the distribution of the forward price, which is conditionally normal. Conditional first moments of the forward can easily be calculated from Equation 6.2:

$$\mathbb{E}\left[F_{\mathcal{F}}^{\mathbb{P}}(T,T_1,T_2) \mid \mathcal{F}_t\right] = F_{\mathcal{F}}^{\mathbb{P}}(t,T_1,T_2) \tag{6.8}$$

Furthermore, we can use Itō's isometry (cf. Jeanblanc et al. (2009, Proposition 1.5.1.1) or Revuz and Yor (1991, Chapter IV)) to calculate the variance of the forward price under $(\mathcal{F}, \mathbb{P})$:

$$\begin{aligned}
\Sigma^2(t,T,T_1,T_2) &= Var\left(F_{\mathcal{F}}^{\mathbb{P}}(T,T_1,T_2) \mid \mathcal{F}_t\right) \\
&= \frac{1}{(T_2-T_1)^2}\,\sigma^2 \int_t^T \bar{\alpha}^2(t,T_1,T_2)ds \\
&= \frac{\sigma^2}{(T_2-T_1)^2}\,\frac{1}{\alpha^2}\int_t^T (e^{-2\alpha(T_2-s)} - e^{-\alpha(T_2-s)}e^{-\alpha(T_1-s)} + e^{-2\alpha(T_1-s)})ds \\
&= \frac{\sigma^2}{(T_2-T_1)^2}\,\frac{1}{\alpha^2}\left(\frac{1}{2\alpha}\left(e^{-2\alpha(T_2-T)} - e^{-2\alpha(T_2-t)} + e^{-2\alpha(T_1-T)} - e^{-2\alpha(T_1-t)}\right)\right. \\
&\quad \left. - 2\frac{1}{2\alpha}\left(e^{-\alpha(T_2+T_1-2T)} - e^{-\alpha(T_2+T_1-2t)}\right)\right) \\
&= \frac{\sigma^2}{(T_2-T_1)^2}\,\frac{1}{\alpha^3}\left(\frac{1}{2}\left(e^{-2\alpha(T_2-T)} - e^{-2\alpha(T_2-t)} + e^{-2\alpha(T_1-T)} - e^{-2\alpha(T_1-t)}\right)\right. \\
&\quad \left. - \left(e^{-\alpha(T_2+T_1-2T)} - e^{-\alpha(T_2+T_1-2t)}\right)\right)
\end{aligned} \tag{6.9}$$

The option price is given by the risk-neutral valuation formula presented for example in Bingham and Kiesel (2004, Theorem 6.1.4). For the sake of simplicity we will assume a zero risk-free interest rate $r = 0$ in the following. Hence:

$$C_{\mathcal{F}}^{\mathbb{P}}\left(t,T,F_{\mathcal{F}}^{\mathbb{P}}(T,T_1,T_2,K)\right) = \mathbb{E}^{\mathbb{P}}\left[(F_{\mathcal{F}}^{\mathbb{P}}(T,T_1,T_2) - K)^+ \mid \mathcal{F}_t\right] \tag{6.10}$$

Remember that the risk-neutral measure was chosen to be the real-world measure $\mathbb{P}$ because the forward already was a martingale under this measure. Introducing the auxiliary function

$$d_1^{\mathcal{F}} = \frac{F_{\mathcal{F}}^{\mathbb{P}}(t,T_1,T_2) - K}{\Sigma(t,T,T_1,T_2)} \tag{6.11}$$

as well as a standard normal random variable Z, we rearrange Equation 6.2:

$$\mathbb{E}\left[\left(F_{\mathcal{F}}^{\mathbb{P}}(T,T_1,T_2) - K\right)^+ \mid \mathcal{F}_t\right] = \mathbb{E}\left[\left(F_{\mathcal{F}}^{\mathbb{P}}(t,T_1,T_2) - K + \Sigma(t,T,T_1,T_2)Z\right)^+ \mid \mathcal{F}_t\right]$$

Thus, we are in the classical (Gaussian) Bachelier setup going back to the foundations of Financial Mathematics in Bachelier (1900). We state:

Proposition 6.4.1. Traditional Call option price. *The price at t of a vanilla call option with maturity T and strike K under the historical filtration $\mathcal{F}_t$ on an electricity forward priced under the historical filtration $\mathcal{F}_t$ and with delivery period in $[T_1, T_2]$ is given by*

$$C^{\mathbb{P}}_{\mathcal{F}}\left(t, T, F^{\mathbb{P}}_{\mathcal{F}}(T, T_1, T_2, K)\right) = \left(F^{\mathbb{P}}_{\mathcal{F}}(t, T_1, T_2) - K\right)\Phi\left(d^{\mathcal{F}}_1\right) + \Sigma\phi\left(d^{\mathcal{F}}_1\right)$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the standard-normal density and distribution and $d^{\mathcal{F}}_1$ is defined as in Equation 6.11.

Proof. This is a straightforward calculation:

$$\begin{aligned}\mathbb{E}&\left[\left(F^{\mathbb{P}}_{\mathcal{F}}(t, T_1, T_2) - K + \Sigma(t, T, T_1, T_2)Z\right)^+ \mid \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\left(F^{\mathbb{P}}_{\mathcal{F}}(t, T_1, T_2) - K\right)\mathbb{1}_{Z>-d^{\mathcal{F}}_1} + \Sigma(t, T, T_1, T_2)Z\mathbb{1}_{Z>-d^{\mathcal{F}}_1} \mid \mathcal{F}_t\right]\end{aligned}$$

and writing the second term in the expectation as an integral immediately yields the result. □

Having remembered the traditional case we can now go on and introduce additional market information.

6.4.2. The Price of a Call with Additional Market Information

Here, we will calculate the option price for the case that both option and forward are priced under the market filtration $\mathcal{G}_t$, i.e. the forward model is $(\mathcal{G}, \mathbb{Q}_{\mathcal{G}})$. Obviously, the forward as given by Equation 6.7 is conditionally normally distributed and the first two moments are given by:

$$\mathbb{E}\left[F^{\mathbb{Q}_{\mathcal{G}}}_{\mathcal{G}}(T, T_1, T_2) \mid \mathcal{G}_t\right] = F^{\mathbb{Q}_{\mathcal{G}}}_{\mathcal{G}}(t, T_1, T_2) \tag{6.12}$$

$$Var\left(F^{\mathbb{Q}_{\mathcal{G}}}_{\mathcal{G}}(T, T_1, T_2) \mid \mathcal{G}_t\right) = \Sigma^2(t, T, T_1, T_2) \tag{6.13}$$

Thus, we have the same variance as in Section 6.4.1 and the risk-neutral valuation formula is

$$C^{\mathbb{Q}_{\mathcal{G}}}_{\mathcal{G}}\left(t, T, F^{\mathbb{Q}_{\mathcal{G}}}_{\mathcal{G}}(T, T_1, T_2), K\right) = \mathbb{E}^{\mathbb{Q}_{\mathcal{G}}}\left[\left(F^{\mathbb{Q}_{\mathcal{G}}}_{\mathcal{G}}(T, T_1, T_2) - K\right)^+ \mid \mathcal{G}_t\right] \tag{6.14}$$

We need one more auxiliary variable defined by

$$d^{\mathcal{G}}_1 = \frac{F^{\mathbb{Q}_{\mathcal{G}}}_{\mathcal{G}}(t, T_1, T_2) - K}{\Sigma(t, T, T_1, T_2)} \tag{6.15}$$

We can then state:

Proposition 6.4.2. Call Option price with additional information. *The price at t of a vanilla call option with maturity T and strike K under the market filtration $\mathcal{G}_t$ on an electricity forward priced under market filtration $\mathcal{G}_t$ and with delivery period in $[T_1, T_2]$ is given by*

$$C_{\mathcal{G}}^{\mathbb{Q}_{\mathcal{G}}}\left(t, T, F_{\mathcal{F}}^{\mathbb{Q}_{\mathcal{G}}}(T, T_1, T_2, K)\right) = \left(F_{\mathcal{G}}^{\mathbb{Q}_{\mathcal{G}}}(t, T_1, T_2) - K\right)\Phi\left(d_1^{\mathcal{G}}\right) + \Sigma\phi\left(d_1^{\mathcal{G}}\right)$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the standard-normal density and distribution and $d_1^{\mathcal{G}}$ is defined as in equation Equation 6.15.

Proof. As in Proposition 6.4.2. □

We remark that the pricing formulae of Proposition 6.4.1 and Proposition 6.4.2 are identical but for the initial forward prices. The prices differ depending on which model was used to calculate them. We see that a trader ignoring future information when pricing a forward would also produce other option prices.

6.4.3. Mixed Cases

Until now, we have considered the same filtration or model to price the forward and the option. For the sake of completeness we will now briefly discuss mixed cases.

If the forward is priced under the historical filtration $\mathcal{F}_t$ and the option under the market filtration $\mathcal{G}_t$, then we need to transform the dynamics from Equation 6.1 and we have already seen that under $\mathbb{Q}_{\mathcal{G}}$ they remain a martingale. Thus, we will end up with the option price as given by Proposition 6.4.2. In other words, even if the underlying forward was priced without additional knowledge, this knowledge is priced in when evaluating the option.

If we now want to calculate the price of an $\mathcal{F}_t$-vanilla call option on a forward evaluated under the market filtration $\mathcal{G}_t$, exactly the opposite happens. Extra information is orthogonal to the space spanned by the historical filtration (cf. Lemma 4.2.2) and thus ignored. We are in the situation of Proposition 6.4.1, again.

6.5. Stylised Examples and Discussion

In this section we illustrate and discuss the findings of this chapter. As mentioned in Section 6.4, we will assume that the spot satisfies $S_t = \mu + X_t$ (for some constant μ), i.e. that we have a constant seasonality (this will bring about more realistic prices). Furthermore, we will assume that market agents are given non-precise future spot information about X_{T_Υ} meaning that we know the value of $\mathbb{E}[X_{T_\Upsilon}|\mathcal{G}_t]$. Their values will be similar to those fitted to market data. Also, for the time axis our setup will be as follows: today is $t = 10$ and we are considering a European call option with maturity $T = 15$ and delivery in $[20, 30]$. Additional information will be provided to the market about the spot at time $T_\Upsilon = 25$ and the seasonality will take value $\mu = 30$.

For different values of the volatility and the speed of mean-reversion Figure 6.5.2 illustrates the value of a traditional at-the-money European call option, i.e. $K = 30$, as calculated in Proposition 6.4.1. With most combinations of parameter values this option has practically zero value, the reason being the averaging effect of the delivery period. For a very low rate of mean-reversion and large volatility we find a positive value for this option.

Figure 6.5.3 shows the corresponding picture for the at-the-money option with additional information as calculated in Proposition 6.4.2. The additional information has expected value $\mathbb{E}[X_{25}|\mathcal{G}_{10}] = 5$. Not surprisingly, the value of the option has a non-zero positive value for all combinations of α and σ. We observe the same effect as above, i.e. larger option prices for larger volatility and small speed of mean-reversion.

An example of an in-the-money option in the traditional pricing framework is illustrated in Figure 6.5.4, where we now assume a strike of $K = 25$. This results in an almost flat price at level 5, as expected. Only for very small speeds of mean-reversion and large volatility does the price increase. Figure 6.5.5 consequently is a plot of the in-the-money call with additional market information (again, as in Proposition 6.4.2). Here, we have changed the sign of the additional information, $\mathbb{E}[X_{25}|\mathcal{G}_{10}] = -5$. The price of the option is generally lower than the one of Figure 6.5.4 and decreasing with decreasing speed of mean-reversion. This is due to the lesser significance of the future information for a higher degree of mean-reversion. But the most striking feature is the fact that the option price increases again for very small α and large σ. In that case the volatility of the spot price is no longer significantly dampened by the mean-reversion of X_t and it is well known that higher volatility causes higher option prices. Hence, there are two forces effecting the option price for small speed of mean-reversion α.

6.6. Contribution and Discussion

This chapter investigates the important aspect of pricing options in a market governed by our new spot-forward relationship. We recall the result from the insider literature about the equivalence of options prices for both the ordinary and the insider trader and provide mathematical and intuitive justifications. Furthermore, we discuss how and if this result translates to electricity markets and in particular find that the classification of what is a fundamental asset in such markets requires consideration. The main result of this chapter is that option prices need not coincide under the historical and the market filtrations as no suitable equivalent replicating strategy is available. This is due to the non-storability of the underlying. Moreover, for vanilla options on forward contracts we deduce closed form pricing formulae. These do not differ with respect to the volatility structure (which is unchanged by enlargement of filtration) but rather with respect to the initial forward price inserted. The stylised examples we provide and their corresponding interpretation make sense economically and show the desired properties and sensitivities.

Summarising, this chapter demonstrates that the information approach we advocate and justify throughout this thesis does influence the pricing of options and derivatives but still allows for closed-form solutions.

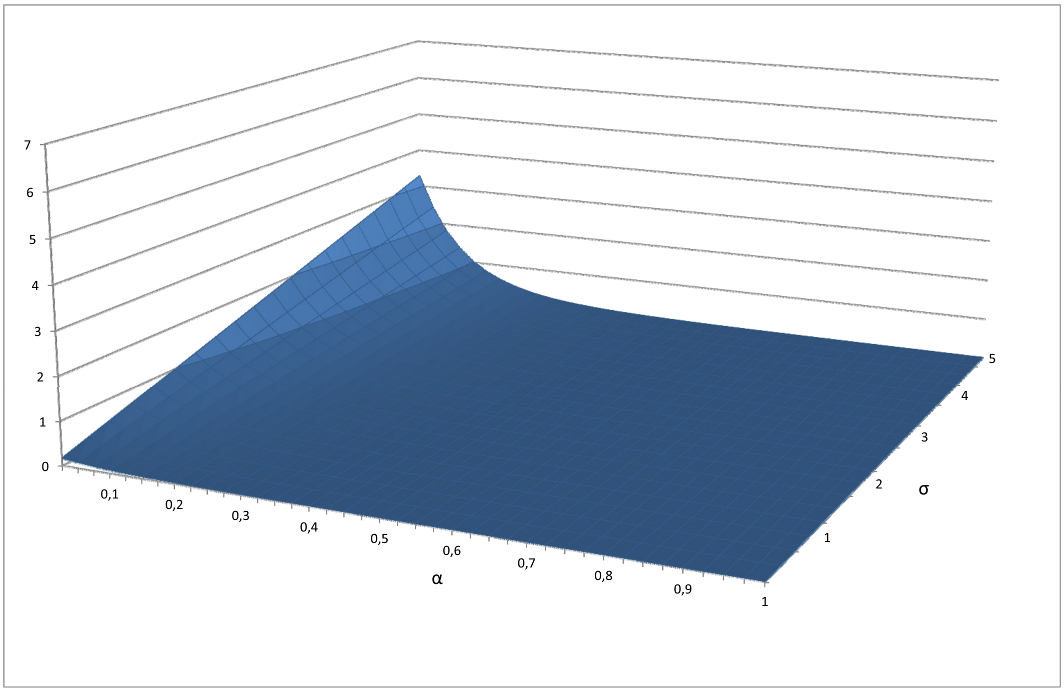


Figure 6.5.2.: At-the-money vanilla call option: Traditional. For parameters: $t = 10$, $T = 15$, $T_1 = 20$, $T_2 = 30$, $X_{10} = 0$, $\mu = 30$ and $K = 30$.

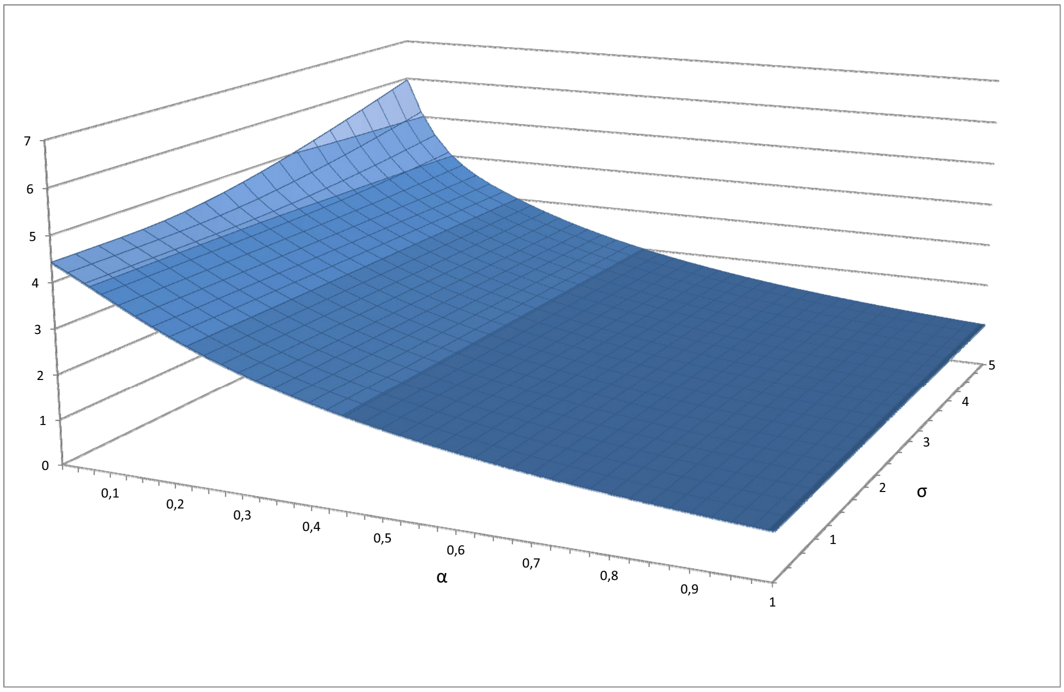


Figure 6.5.3.: At-the-money vanilla call option: Additional information. For $t = 10$, $T = 15$, $T_1 = 20$, $T_2 = 30$, $X_{10} = 0$, $\mu = 30$, $K = 30$, $T_\Upsilon = 25$, $\mathbb{E}[X_{25} \mid \mathcal{G}_{10}] = 5$.

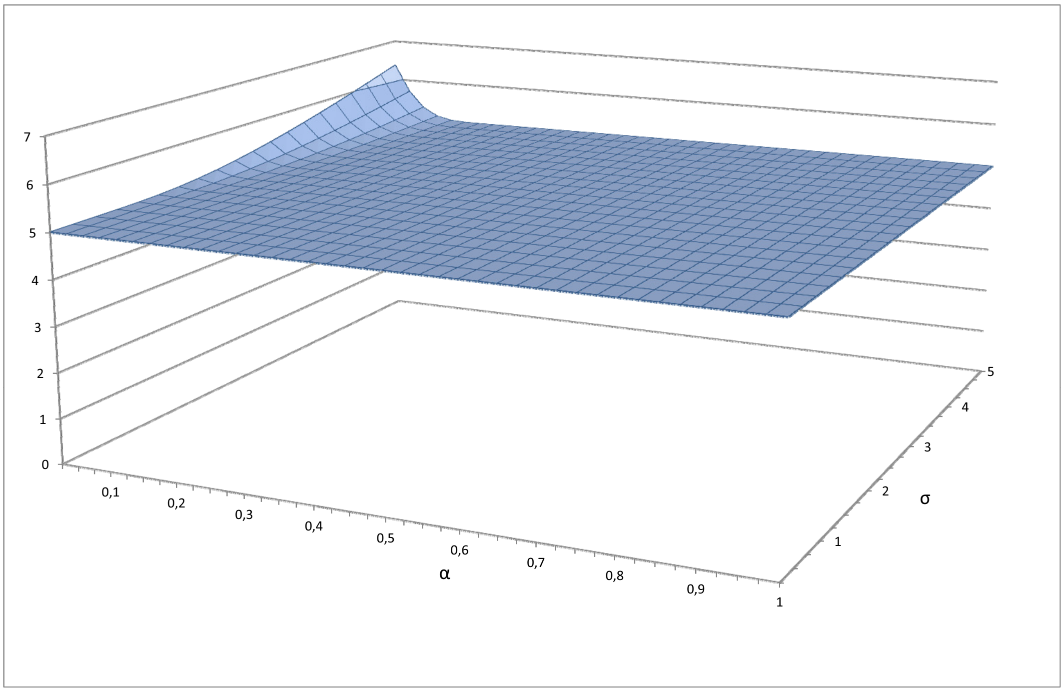


Figure 6.5.4.: In-the-money vanilla call option: Traditional. For parameters: $t = 10$, $T = 15$, $T_1 = 20$, $T_2 = 30$, $X_{10} = 0$, $\mu = 30$ and $K = 25$.

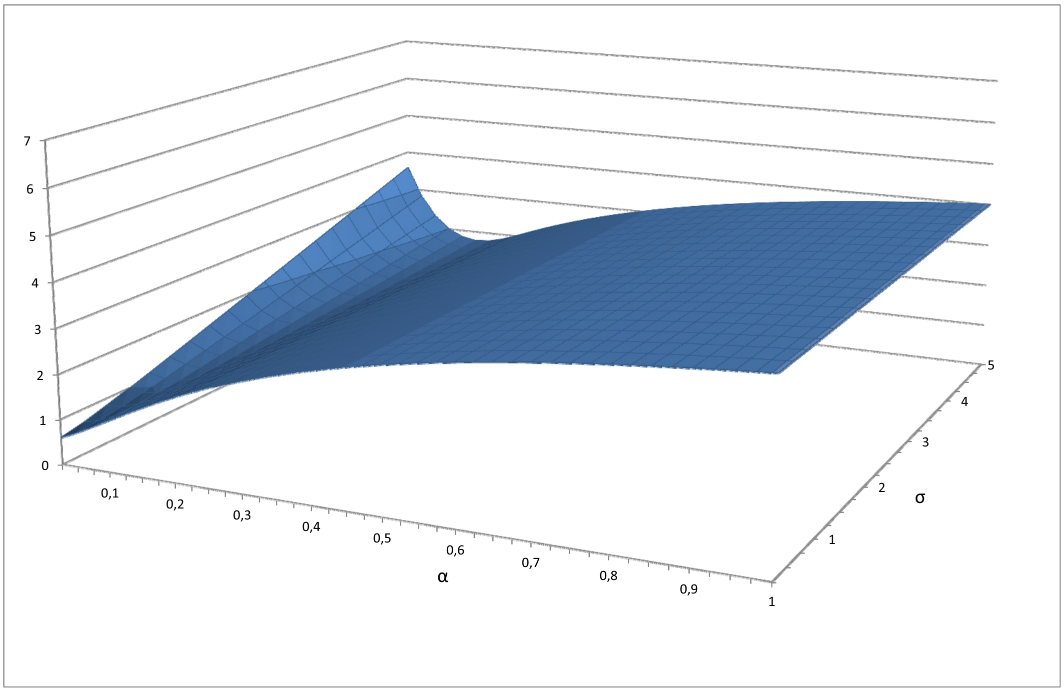


Figure 6.5.5.: In-the-money vanilla call option: Additional information. For $t = 10$, $T = 15$, $T_1 = 20$, $T_2 = 30$, $X_{10} = 0$, $\mu = 30$, $K = 25$, $T_\Upsilon = 25$, $\mathbb{E}[X_{25} \mid \mathcal{G}_{10}] = -5$.

Chapter 7.

Information Premium and Market Power

7.1. Literature Overview and Summary

We have discussed above that electricity markets are incomplete. Incomplete markets do not feature a unique arbitrage-free pricing measure (this is the second fundamental theorem of Financial Mathematics, Bingham and Kiesel (2004, Theorem 4.3.1)). Thus, there is a multitude of consistent prices available for derivatives such as, for example, forwards. This suggests to apply the theory of indifference pricing. Introducing a utility function we can find the price that makes the investor under consideration indifferent between the utility of possessing the derivative and not possessing it. This price is then called the indifference price of the derivative. We refer to Carmona (2009) (and in particular Chapter 2 therein) for a detailed discussion of indifference pricing.

For electricity markets, this is exactly the framework presented in Benth, Cartea, and Kiesel (2008a) which tries to find an explanation for the shape of the risk premium (see Proposition 4.2.1) in terms of market power and the risk appetite of different types of market agents. In this chapter we will provide some ideas of how one can introduce our information approach to the ideas of that paper. Mathematically, we will show how one can use the results from Section 2.2.4 to explicitly calculate indifference forward prices for the exponential utility of Benth et al. (2008a) and the arithmetic spot model as introduced in Chapter 3. To the best of our knowledge this is the first time that a calculation like this is proposed other than with the (cancelling) combination of a logarithmic utility and an exponential underlying (as in the literature on insider trading mentioned in Chapter 6).

This chapter will be structured as follows: to begin with, we will summarise very briefly the approach and ideas of Benth et al. (2008a). Then, with this framework in mind, we will present our calculations in Section 7.3. We will check plausibility with some stylised examples. Finally, we will conclude in Section 7.4.

As mentioned in Section 1.4.2, this chapter is the basis of a section of a working paper.

7.2. The Model of Benth/Cartea/Kiesel and Framework

The authors of Benth et al. (2008a) commence their study by setting up the spot model we have introduced in Chapter 3. Furthermore, they use the exponential utility function with constant risk-aversion parameter γ. This is given by

$$U(x) = 1 - \exp(-\gamma x) \tag{7.1}$$

They differentiate between the risk-aversion of a producer γ_P and that of a retailer γ_R. For a future time interval $[T_1, T_2]$ market agents then have two alternatives: either they invest in a forward with that delivery period, or they buy or sell on the spot market. The forward price making the producer indifferent between the two alternatives is then given by the solution of

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(-\gamma_P\int_{T_1}^{T_2} S_u du\right)|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\gamma_P(T_2-T_1)F^P_{\mathcal{F}}(t,T_1,T_2)\right)|\mathcal{F}_t\right]$$

For simplicity we will assume $t < T_1 < T_2$ in this chapter but clearly all other arrangements of the time axis are easily dealt with. Now, rearranging terms, Benth et al. (2008a, Equation (2.9)) provide the indifference forward prices

$$F^P_{\mathcal{F}}(t,T_1,T_2) = -\gamma_P\frac{1}{T_2-T_1}\log\left(\mathbb{E}^{\mathbb{P}}\left[\exp\left(-\gamma_P\int_{T_1}^{T_2} S_u du\right)|\mathcal{F}_t\right]\right) \tag{7.2}$$

for producers and

$$F^R_{\mathcal{F}}(t,T_1,T_2) = \gamma_R\frac{1}{T_2-T_1}\log\left(\mathbb{E}^{\mathbb{P}}\left[\exp\left(\gamma_R\int_{T_1}^{T_2} S_u du\right)|\mathcal{F}_t\right]\right) \tag{7.3}$$

for retailers. Benth et al. (2008a, Proposition 2.1) then find the solution of these formulae with much the same methodology as in Proposition 3.4.3 of this thesis:

Proposition 7.2.1. Indifference forward price of a producer. *The producer's indifference price of a forward contract in t with delivery period $[T_1, T_2]$ is given by*

$$\begin{aligned}F^P_{\mathcal{F}}(t,T_1,T_2) = &\frac{1}{T_2-T_1}\left(\int_{T_1}^{T_2}\Lambda_u du + \bar{\alpha}(t,T_1,T_2)X_t + \bar{\beta}(t,T_1,T_2)Y_t\right)\\ &-\frac{\gamma_P}{2}\frac{1}{T_2-T_1}\int_t^{T_2}\sigma^2\bar{\alpha}(s,T_1,T_2)ds\\ &-\frac{1}{\gamma_P}\frac{1}{T_2-T_1}\int_t^{T_2}\phi(-\gamma_P\bar{\beta}(s,T_1,T_2))ds\end{aligned}$$

where functions $\bar{\alpha}(\cdot)$ and $\bar{\beta}(\cdot)$ are (as usually) given by Notation 3.4.1.

The corresponding retailer price can easily be calculated along the same lines. Next, the authors continue to introduce the so-called market power function which takes value in $[0, 1]$, $p(t, T_1, T_2) = 1$ if the producer and $p(t, T_1, T_2) = 0$ if the retailer has full market power. The market price of the forward is then consequently defined as a linear combination:

$$F(t,T_1,T_2) = p(t,T_1,T_2)F^R_{\mathcal{F}}(t,T_1,T_2) + (1-p(t,T_1,T_2))\,F^P_{\mathcal{F}}(t,T_1,T_2) \tag{7.4}$$

Furthermore, according to the traditional spot-forward relationship Proposition 4.2.1 the forward price is given by $F^{\mathbb{Q}}_{\mathcal{F}}(t, T_1, T_2)$. Choosing a parametric pricing measure $\mathbb{Q}$ now suggests two courses of action: either we measure, in some way, the market power of players and find corresponding measure change parameters or we find some criterion for the measure change (distance-minimising for example) and deduce the matching market power. Benth et al. (2008a) follow both approaches and also conduct an empirical study with EEX data. Their study confirms the intuition that market retailers are more powerful for far-out deliveries whereas producers have a larger power for forward contracts close to expiry. On the one hand producers need to hedge their large, long-lasting investments. On the other hand retailers (having guaranteed fixed prices to their customers) are afraid of running into spot spikes. In the empirical studies of Chapter 8 of this thesis we will reproduce exactly this behaviour for the emissions dataset (as introduced in Section 1.2.1) but we will also see that for the newer Moratorium dataset (cf. Section 1.2.2) this is not the case. The market seems to have matured over the course of the years, in particular the number of positive jumps has drastically decreased.

7.3. Indifference Forward Price with Future Information

We have seen that Benth, Cartea, and Kiesel (2008a) use the traditional spot-forward relationship. We will now examine how indifference prices look and behave like in the presence of additional future information, i.e. with respect to the new spot-forward relationship Proposition 4.2.2. To this end, we let, as usual, $\mathcal{G}_t$ be the (enlarged) market filtration, assumed to satisfy

$$\mathcal{G}_t = \mathcal{H}_t = \mathcal{F}_t \vee \sigma(X_{T_\Upsilon})$$

This is the case known from Section 5.2.3 and with precise future information. The indifference forward price of the producer is then given as the solution of

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(-\gamma_P \int_{T_1}^{T_2} S_u du\right) |\mathcal{G}_t\right] = \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\gamma_P (T_2 - T_1) F_{\mathcal{G}}^P(t, T_1, T_2)\right) |\mathcal{G}_t\right]$$

Rearranging terms the expression becomes

$$F_{\mathcal{G}}^P(t, T_1, T_2) = -\gamma_P \frac{1}{T_2 - T_1} \log\left(\mathbb{E}^{\mathbb{P}}\left[\exp\left(-\gamma_P \int_{T_1}^{T_2} S_u du\right) |\mathcal{G}_t\right]\right) \tag{7.5}$$

We now begin to calculate the solution of Equation 7.5:

$$\begin{aligned} F_{\mathcal{G}}^P(t, T_1, T_2) = & \frac{1}{T_2 - T_1}\left(\int_{T_1}^{T_2} \Lambda_u du + \bar{\alpha}(t, T_1, T_2) X_t + \bar{\beta}(t, T_1, T_2) Y_t\right) \\ & - \frac{1}{\gamma_P} \frac{1}{T_2 - T_1} \log\left(\mathbb{E}^{\mathbb{P}}\left[\exp\left(-\gamma_P \int_{T_1}^{T_2} \int_t^u \sigma e^{-\alpha(u-s)} dW_s du\right) |\mathcal{G}_t\right]\right) \\ & - \frac{1}{\gamma_P} \frac{1}{T_2 - T_1} \int_t^{T_2} \phi(-\gamma_P \bar{\beta}(s, T_1, T_2)) ds \end{aligned} \tag{7.6}$$

As usual, we have used the fact that Λ_t is deterministic and that X_t and Y_t are $\mathcal{G}_t$-measurable. Also, for L_t, both filtrations coincide. This leaves the Brownian term to be calculated. But what is the expectation under the market filtration of a functional of the Gaussian Ornstein-Uhlenbeck process?

7.3.1. The Distribution of the Exponential Brownian Integral

We will now evaluate the Brownian term of Equation 7.6. We are facing the expectation of the exponential of an integral with respect to the $\mathcal{F}_t$-Brownian motion W_t. We have seen in Section 5.2 that the $\mathcal{G}_t$-decomposition of this integral is (ignoring the utility parameter γ for now)

$$\begin{aligned}\int_{T_1}^{T_2}\int_t^u \sigma e^{-\alpha(u-s)} dW_s du \\ = \int_{T_1}^{T_2}\int_t^u \sigma e^{-\alpha(u-s)} d\xi_s du + \int_{T_1}^{T_2}\int_t^u \sigma e^{-\alpha(u-s)} \mu_s^{\mathcal{G}} ds du \qquad (7.7)\end{aligned}$$

As in the previous chapters, ξ_t is a $\mathcal{G}_t$-Brownian motion and the $\mathcal{G}_t$-measurable process $\mu_t^{\mathcal{G}}$ is given by

$$\mu_t^{\mathcal{G}} = a(t)\left(\int_t^{T_\Upsilon} e^{\alpha u} dW_u\right)$$

with the auxiliary function $a(\cdot)$ as in Equation 5.10:

$$a(t) = \frac{2\alpha e^{\alpha t}}{e^{2\alpha T_\Upsilon} - e^{2\alpha t}}$$

In order to get a closed-form solution of the expectation we need to know the properties (and distribution) of this $\mathcal{G}_t$-process. In Benth et al. (2008a) the authors use the properties of the log-normal distribution because they work under the historical filtration. Here, we have a $\mathcal{G}_t$-Brownian motion ξ_t but also a random drift involving $\mu_t^{\mathcal{G}}$ and thus W_t.

The basic idea in this chapter will be to view Equation 7.7 not so much as a decomposition (as before in this thesis) but rather as a stochastic differential equation (as in Section 2.2.4, the mathematical background of this chapter).

We begin calculations by plugging in the information yield and by concentrating on terms in s of Equation 7.7 only. This means we consider significant terms of the inner integral:

$$\begin{aligned}\int_t^u e^{\alpha s} dW_s &= \int_t^u e^{\alpha s} d\xi_s + \int_t^u e^{\alpha s} a(s)\left(\int_s^{T_\Upsilon} e^{\alpha v} dW_v\right) ds \\ &= \int_t^u e^{\alpha s} d\xi_s + \int_t^u e^{\alpha s} a(s) ds \left(\int_0^{T_\Upsilon} e^{\alpha v} dW_v\right) \\ &\qquad - \int_t^u e^{\alpha s} a(s)\left(\int_0^s e^{\alpha v} dW_v\right) ds \qquad (7.8)\end{aligned}$$

We are going to continue with the dynamics of Equation 7.8

$$e^{\alpha s} dW_s = e^{\alpha s} d\xi_s + e^{\alpha s} a(s)\left(\int_0^{T_\Upsilon} e^{\alpha v} dW_v\right) ds - e^{\alpha s} a(s)\left(\int_0^s e^{\alpha v} dW_v\right) ds \quad (7.9)$$

Let us define an auxiliary process

$$\chi_s = \int_0^s e^{\alpha v} dW_v \tag{7.10}$$

and corresponding dynamics $d\chi_s = e^{\alpha s} dW_s$. Then we can rewrite the stochastic differential Equation 7.9 in the following way:

$$d\chi_s = e^{\alpha s} d\xi_s + e^{\alpha s} a(s)\chi_{T_\Upsilon} ds - e^{\alpha s} a(s)\chi_s ds \tag{7.11}$$

But we know the precise value of χ_{T_Υ} and thus this is a linear stochastic differential equation in s. Remembering Section 2.2.4 we will try to solve it using standard methods. Firstly, we identify the deterministic and homogeneous version of the stochastic differential equation

$$\frac{d\tilde{\chi}(s)}{ds} = -e^{\alpha s} a(s)\tilde{\chi}(s) \tag{7.12}$$

This can be solved via

$$\begin{aligned} d(\log(\tilde{\chi}(s))) &= -e^{\alpha s} a(s) ds \\ \int_0^u d(\log(\tilde{\chi}(s))) &= -\int_0^u e^{\alpha s} \frac{2\alpha e^{\alpha s}}{e^{2\alpha T_\Upsilon} - e^{2\alpha s}} ds \end{aligned}$$

Substituting $x = e^{2\alpha s}$ we can easily solve the integral:

$$\tilde{\chi}(u) = \frac{e^{2\alpha T_\Upsilon} - e^{2\alpha u}}{e^{2\alpha T_\Upsilon} - 1} \tag{7.13}$$

The multiplicative inverse of function $\tilde{\chi}(u)$ is

$$\tilde{\chi}^{-1}(u) = \frac{e^{2\alpha T_\Upsilon} - 1}{e^{2\alpha T_\Upsilon} - e^{2\alpha u}} \tag{7.14}$$

In order to solve the stochastic differential Equation 7.11 we are now going to apply Itō's lemma to function $f(s,x)$ defined by

$$f(s,x) = \tilde{\chi}^{-1}(s)x \ , \ f_s(s,x) = e^{\alpha s} a(s) f(s,x) \ , \ f_x(s,x) = \tilde{\chi}^{-1}(s) \ , \ f_{xx}(s,x) = 0$$

The process x under consideration will be χ_s of Equation 7.11 with expectation $e^{\alpha s} a(s)\chi_{T_\Upsilon} - e^{\alpha s} a(s)\chi_s$ and variance $e^{\alpha s}$. Thus,

$$\begin{aligned} &d(\tilde{\chi}^{-1}(s)\chi_s) \\ &= \left(e^{\alpha s} a(s)\tilde{\chi}^{-1}(s)\chi_s + \left(e^{\alpha s} a(s)\chi_{T_\Upsilon} - e^{\alpha s} a(s)\chi_s\right)\tilde{\chi}^{-1}(s)\right) ds + e^{\alpha s}\tilde{\chi}^{-1}(s) d\xi_s \\ &= \tilde{\chi}^{-1}(s) e^{\alpha s} a(s)\chi_{T_\Upsilon} ds + \tilde{\chi}^{-1}(s) e^{\alpha s} d\xi_s \end{aligned} \tag{7.15}$$

We integrate Equation 7.15 from t to u:

$$\int_t^u d(\tilde{\chi}^{-1}(s)\chi_s) = \int_t^u \tilde{\chi}^{-1}(s) e^{\alpha s} a(s)\chi_{T_\Upsilon} ds + \int_t^u \tilde{\chi}^{-1}(s) e^{\alpha s} d\xi_s \tag{7.16}$$

Defining the auxiliary function

$$C(t,u) = \frac{e^{2\alpha T_\Upsilon} - e^{2\alpha u}}{e^{2\alpha T_\Upsilon} - e^{2\alpha t}} \tag{7.17}$$

we simplify Equation 7.16 and get

$$\chi_u = C(t,u)\chi_t + \int_t^u C(s,u)e^{\alpha s}a(s)ds\ \chi_{T_\Upsilon} + \int_t^u C(s,u)e^{\alpha s}d\xi_s \tag{7.18}$$

We remark that $\tilde{\chi}(u) = C(0,u)$. Also, with the same substitution as before, we can simplify the middle (deterministic) integral yielding

$$\chi_u = C(t,u)\chi_t + (1 - C(t,u))\ \chi_{T_\Upsilon} + \int_t^u C(s,u)e^{\alpha s}d\xi_s \tag{7.19}$$

As postulated for the Brownian bridge in Section 2.2.4 this shows that χ_u is conditionally normally distributed under the enlarged market filtration. Going back to our original problem this knowledge will finally allow us to use the rules of the expectation of a log-normal variable.

We can now rearrange Equation 7.7

$$\begin{aligned}\int_{T_1}^{T_2}\int_t^u \sigma e^{-\alpha(u-s)}dW_s du &= \int_{T_1}^{T_2}\sigma e^{-\alpha u}\int_t^u e^{\alpha s}dW_s du = \int_{T_1}^{T_2}\sigma e^{-\alpha u}\int_t^u d\chi_s du\\ &= \int_{T_1}^{T_2}\sigma e^{-\alpha u}(\chi_u - \chi_t)du \end{aligned}\tag{7.20}$$

and then plug in Equation 7.19 yielding

$$\begin{aligned}&\int_{T_1}^{T_2}\int_t^u \sigma e^{-\alpha(u-s)}dW_s du\\ &= \int_{T_1}^{T_2}\sigma e^{-\alpha u}(1 - C(t,u))du(\chi_{T_\Upsilon} - \chi_t) + \int_{T_1}^{T_2}\int_t^u \sigma e^{-\alpha(u-s)}C(s,u)d\xi_s du\end{aligned} \tag{7.21}$$

And this representation now enables us to calculate the expectation of the Brownian part of Equation 7.5.

7.3.2. The Expectation of the Exponential Brownian Integral

Now, we want to use the fact that the expectation of a log-normal variable can be expressed in terms of the mean and variance of the normally distributed exponent. Thus, we need to calculate the first two moments of Equation 7.21. The expectation is:

$$
\begin{aligned}
&\mathbb{E}\left[\int_{T_1}^{T_2}\int_t^u \sigma e^{-\alpha(u-s)} dW_s du \mid \mathcal{G}_t\right] \\
&= \mathbb{E}\left[\int_{T_1}^{T_2} \sigma e^{-\alpha u}\left(1 - C(t,u)\right) du \left(\chi_{T_\Upsilon} - \chi_t\right) \mid \mathcal{G}_t\right] \\
&= \int_{T_1}^{T_2} \sigma e^{-\alpha u}\left(1 - \frac{e^{2\alpha T_\Upsilon} - e^{2\alpha u}}{e^{2\alpha T_\Upsilon} - e^{2\alpha t}}\right) du \int_t^{T_\Upsilon} e^{\alpha s} dW_s \\
&= \sigma \int_{T_1}^{T_2} e^{-\alpha u}(e^{2\alpha T_\Upsilon} - e^{2\alpha u} - e^{2\alpha T_\Upsilon} + e^{2\alpha t}) du \, \frac{1}{e^{2\alpha T_\Upsilon} - e^{2\alpha t}} \int_t^{T_\Upsilon} e^{\alpha s} dW_s \\
&= (T_2 - T_1) I_{\mathcal{G}}(t, T_1, T_2; T_\Upsilon)
\end{aligned}
\tag{7.22}
$$

Here, in the last step, we have straightforwardly solved the integral. Clearly, Equation 7.22 is our well-known expression of the information premium as calculated in Proposition 5.3.4. In particular, this confirms our calculations from Section 7.3.1.

The new term appearing in the indifference price will be the variance of Equation 7.21. In order to calculate this variance, one alternative is to use Itō's isometry which requires interchanging the order of integration of the double integral:

$$
\int_{T_1}^{T_2}\int_t^u \sigma e^{-\alpha(u-s)} C(s,u) d\xi_s du = \int_t^{T_2}\int_{\max\{s,T_1\}}^{T_2} \sigma e^{-\alpha(u-s)} C(s,u) du d\xi_s \tag{7.23}
$$

We can now solve the inner (deterministic) integral:

$$
\begin{aligned}
&\int_t^{T_2}\int_{\max\{s,T_1\}}^{T_2} \sigma e^{-\alpha(u-s)} C(s,u) du d\xi_s \\
&= \int_t^{T_2} \frac{\sigma e^{\alpha s}}{e^{2\alpha T_\Upsilon} - e^{2\alpha s}} \int_{\max\{s,T_1\}}^{T_2} \left(e^{-\alpha u} e^{2\alpha T_\Upsilon} - e^{\alpha u}\right) du d\xi_s \\
&= \int_t^{T_1} \frac{\sigma e^{\alpha s}}{e^{2\alpha T_\Upsilon} - e^{2\alpha s}} \left(-\frac{1}{\alpha}\left(e^{2\alpha T_\Upsilon}\left(e^{-\alpha T_2} - e^{-\alpha T_1}\right) - \left(e^{\alpha T_2} - e^{\alpha T_1}\right)\right)\right) d\xi_s \\
&+ \int_{T_1}^{T_2} \frac{\sigma e^{\alpha s}}{e^{2\alpha T_\Upsilon} - e^{2\alpha s}} \left(-\frac{1}{\alpha}\left(e^{2\alpha T_\Upsilon}\left(e^{-\alpha T_2} - e^{-\alpha s}\right) - \left(e^{\alpha T_2} - e^{\alpha s}\right)\right)\right) d\xi_s
\end{aligned}
\tag{7.24}
$$

The intervals of the integrals do not overlap. Hence, the overall variance is just the sum of the variances. Now, applying Itō's isometry, we calculate the variance of the first integral:

$$
\begin{aligned}
&Var\left(\int_t^{T_1} \frac{\sigma e^{\alpha s}}{e^{2\alpha T_\Upsilon} - e^{2\alpha s}} \left(-\frac{1}{\alpha}\left(e^{2\alpha T_\Upsilon}\left(e^{-\alpha T_2} - e^{-\alpha T_1}\right) - \left(e^{\alpha T_2} - e^{\alpha T_1}\right)\right)\right) d\xi_s \mid \mathcal{G}_t\right) \\
&= \int_t^{T_1} \frac{\sigma^2 e^{2\alpha s}}{(e^{2\alpha T_\Upsilon} - e^{2\alpha s})^2} \left(-\frac{1}{\alpha}\left(e^{2\alpha T_\Upsilon}\left(e^{-\alpha T_2} - e^{-\alpha T_1}\right) - \left(e^{\alpha T_2} - e^{\alpha T_1}\right)\right)\right)^2 ds \\
&= \frac{1}{2}\frac{\sigma^2}{\alpha^3}\left(e^{2\alpha T_\Upsilon}\left(e^{-\alpha T_2} - e^{-\alpha T_1}\right) - \left(e^{\alpha T_2} - e^{\alpha T_1}\right)\right)^2 \\
&\qquad \left(\frac{1}{e^{2\alpha T_\Upsilon} - e^{2\alpha T_1}} - \frac{1}{e^{2\alpha T_\Upsilon} - e^{2\alpha t}}\right)
\end{aligned}
\tag{7.25}
$$

The second integral is somewhat more complicated:

$$
\begin{aligned}
&Var\left(\int_{T_1}^{T_2} \frac{\sigma e^{\alpha s}}{e^{2\alpha T_\Upsilon} - e^{2\alpha s}} \left(-\frac{1}{\alpha}\left(e^{2\alpha T_\Upsilon}\left(e^{-\alpha T_2} - e^{-\alpha s}\right) - \left(e^{\alpha T_2} - e^{\alpha s}\right)\right)\right) d\xi_s \mid \mathcal{G}_t\right) \\
&\quad = \frac{\sigma^2}{\alpha^2}\int_{T_1}^{T_2} \frac{\left(e^{2\alpha T_\Upsilon}\left(e^{-\alpha(T_2-s)} - 1\right) - \left(e^{\alpha(T_2+s)} - e^{2\alpha s}\right)\right)^2}{(e^{2\alpha T_\Upsilon} - e^{2\alpha s})^2} ds \qquad (7.26)
\end{aligned}
$$

To simplify notation we are now going to substitute $x = e^{\alpha s}$ (giving $ds = \frac{1}{\alpha}\frac{1}{x}dx$). Also, we introduce two auxiliary functions $A^2 = e^{2\alpha T_\Upsilon}$ and $B = e^{\alpha T_2}$. Equation 7.26 then becomes:

$$
\begin{aligned}
\ldots &= \frac{\sigma^2}{\alpha^2}\int_{s=T_1}^{s=T_2} \frac{\left(A^2(\frac{1}{B}x - 1) - (Bx - x^2)\right)^2}{(A^2 - x^2)^2}\frac{1}{\alpha}\frac{1}{x}dx \\
&= \frac{\sigma^2}{\alpha^3}\int_{s=T_1}^{s=T_2} \frac{\left((A^2 - x^2) - x(\frac{A^2}{B} - B)\right)^2}{(A^2 - x^2)^2 x}dx \\
&= \frac{\sigma^2}{\alpha^3}\left(\int_{s=T_1}^{s=T_2} \frac{1}{x}dx + \int_{s=T_1}^{s=T_2} \frac{-2(\frac{A^2}{B} - B)}{A^2 - x^2}dx + \int_{s=T_1}^{s=T_2} \frac{x(\frac{A^2}{B} - B)^2}{(A^2 - x^2)^2}dx\right) \qquad (7.27)
\end{aligned}
$$

We can solve the three integrals of Equation 7.27 separately. The first integral yields

$$
\int_{s=T_1}^{s=T_2} \frac{1}{x}dx = [\log e^{\alpha s}]_{T_1}^{T_2} = \alpha T_2 - \alpha T_1 \qquad (7.28)
$$

The second integral can be calculated according to

$$
\begin{aligned}
&\int_{s=T_1}^{s=T_2} \frac{-2(\frac{A^2}{B} - B)}{A^2 - x^2}dx = -2\left(\frac{A^2}{B} - B\right)\int_{s=T_1}^{s=T_2} \frac{1}{A^2 - x^2}dx \\
&\quad = -2\left(\frac{A^2}{B} - B\right)\left[\frac{1}{A}\tanh^{-1}\left(\frac{x}{A}\right)\right]_{s=T_1}^{s=T_2} \\
&\quad = -2\left(\frac{A^2}{B} - B\right)\frac{1}{A}\left[\frac{1}{2}\log\left(\frac{A + x}{A - x}\right)\right]_{s=T_1}^{s=T_2} \\
&\quad = \left(e^{-\alpha(T_\Upsilon - T_2)} - e^{-\alpha(T_2 - T_\Upsilon)}\right)\log\left(\frac{(e^{\alpha T_\Upsilon} + e^{\alpha T_2})(e^{\alpha T_\Upsilon} - e^{\alpha T_1})}{(e^{\alpha T_\Upsilon} - e^{\alpha T_2})(e^{\alpha T_\Upsilon} + e^{\alpha T_1})}\right) \qquad (7.29)
\end{aligned}
$$

The third integral equates to:

$$
\begin{aligned}
&\int_{T_1}^{T_2} \frac{x(\frac{A^2}{B} - B)^2}{(A^2 - x^2)^2}dx = \frac{1}{B^2}(A^2 - B^2)^2\int_{T_1}^{T_2} \frac{x}{(A^2 - x^2)^2}dx \\
&\quad = \frac{1}{B^2}(A^2 - B^2)^2\left[\frac{1}{2}\frac{1}{A^2 - x^2}\right]_{s=T_1}^{s=T_2} \\
&\quad = \frac{1}{2}e^{-2\alpha T_2}\left(e^{2\alpha T_\Upsilon} - e^{2\alpha T_2}\right)^2\left(\frac{1}{e^{2\alpha T_\Upsilon} - e^{2\alpha T_2}} - \frac{1}{e^{2\alpha T_\Upsilon} - e^{2\alpha T_1}}\right) \\
&\quad = \frac{1}{2}e^{-2\alpha T_2}\left((e^{2\alpha T_\Upsilon} - e^{2\alpha T_2}) - \frac{(e^{2\alpha T_\Upsilon} - e^{2\alpha T_2})^2}{e^{2\alpha T_\Upsilon} - e^{2\alpha T_1}}\right) \qquad (7.30)
\end{aligned}
$$

Bringing together Equation 7.28, Equation 7.29 and Equation 7.30 we find the overall variance to be:

$$
\begin{aligned}
&Var\left(\int_{T_1}^{T_2}\int_t^u \sigma e^{-\alpha(u-s)}dW_s du|\mathcal{G}_t\right)\\
&= \frac{1}{2}\frac{\sigma^2}{\alpha^3}\left(e^{2\alpha T_\Upsilon}\left(e^{-\alpha T_2}-e^{-\alpha T_1}\right)-\left(e^{\alpha T_2}-e^{\alpha T_1}\right)\right)^2\left(\frac{1}{e^{2\alpha T_\Upsilon}-e^{2\alpha T_1}}-\frac{1}{e^{2\alpha T_\Upsilon}-e^{2\alpha t}}\right)\\
&+\frac{\sigma^2}{\alpha^3}\left(\left(e^{-\alpha(T_\Upsilon-T_2)}-e^{-\alpha(T_2-T_\Upsilon)}\right)\log\left(\frac{(e^{\alpha T_\Upsilon}+e^{\alpha T_2})(e^{\alpha T_\Upsilon}-e^{\alpha T_1})}{(e^{\alpha T_\Upsilon}-e^{\alpha T_2})(e^{\alpha T_\Upsilon}+e^{\alpha T_1})}\right)\right)\\
&+\frac{\sigma^2}{\alpha^3}\left(\alpha(T_2-T_1)\right)+\frac{1}{2}\frac{\sigma^2}{\alpha^3}e^{-2\alpha T_2}\left((e^{2\alpha T_\Upsilon}-e^{2\alpha T_2})-\frac{(e^{2\alpha T_\Upsilon}-e^{2\alpha T_2})^2}{e^{2\alpha T_\Upsilon}-e^{2\alpha T_1}}\right)
\end{aligned}
\tag{7.31}
$$

To ease notation we introduce:

Notation 7.3.1. Auxiliary function. We define $\breve{\alpha}(t,T_1,T_2;T_\Upsilon)$ to be

$$
\breve{\alpha}(t,T_1,T_2;T_\Upsilon)=\begin{cases}-\frac{1}{\alpha}\frac{e^{\alpha t}}{e^{2\alpha T_\Upsilon}-e^{2\alpha t}}\left(e^{2\alpha T_\Upsilon}\left(e^{-\alpha T_2}-e^{-\alpha T_1}\right)-\left(e^{\alpha T_2}-e^{\alpha T_1}\right)\right) & t\le T_1\\ -\frac{1}{\alpha}\frac{e^{\alpha t}}{e^{2\alpha T_\Upsilon}-e^{2\alpha t}}\left(e^{2\alpha T_\Upsilon}\left(e^{-\alpha T_2}-e^{-\alpha t}\right)-\left(e^{\alpha T_2}-e^{\alpha t}\right)\right) & t> T_1\end{cases}
$$

With this notation we can abbreviate and write:

$$
Var\left(\int_{T_1}^{T_2}\int_t^u \sigma e^{-\alpha(u-s)}dW_s du|\mathcal{G}_t\right)=\int_t^{T_2}\sigma^2\breve{\alpha}^2(s,T_1,T_2;T_\Upsilon)ds
$$

And this finally allows to propagate the following proposition:

Proposition 7.3.1. Indifference price of a producer. *The indifference price in t of a forward with delivery in $[T_1,T_2]$ of a producer having access to precise information about the base component of the spot at time T_Υ is given by*

$$
\begin{aligned}
F_P(t,T_1,T_2) &= -\frac{1}{\gamma_P}\frac{1}{T_2-T_1}\log\left(\mathbb{E}^{\mathbb{P}}\left[\exp\left(-\gamma_P\int_{T_1}^{T_2}(\Lambda_u+X_u+Y_u)du\right)\mid\mathcal{G}_t\right]\right)\\
&=\frac{1}{T_2-T_1}\left(\int_{T_1}^{T_2}\Lambda_u du+\bar{\alpha}(t,T_1,T_2)X_t+\bar{\beta}(t,T_1,T_2)Y_t\right)\\
&\quad-\frac{1}{\gamma_P}\frac{1}{T_2-T_1}\int_t^{T_2}\phi(-\gamma_P\bar{\beta}(s,T_1,T_2))ds\\
&\quad+I_{\mathcal{G}}(t,T_1,T_2;T_\Upsilon)-\frac{\gamma_P}{2}\frac{1}{T_2-T_1}\int_t^{T_2}\sigma^2\breve{\alpha}^2(t,T_1,T_2;T_\Upsilon)ds
\end{aligned}
$$

where the solution to the last term is given by Equation 7.31.

Similarly, we can now readily calculate the retailer's indifference price:

Proposition 7.3.2. Indifference price of a retailer. *The indifference price in t of a forward with delivery in $[T_1,T_2]$ of a retailer having access to precise information* *about the base component of the spot at time T_Υ is given by*

$$
\begin{aligned}
F_R(t,T_1,T_2) &= \frac{1}{\gamma_R}\frac{1}{T_2-T_1}\log\left(\mathbb{E}^{\mathbb{P}}\left[\exp\left(\gamma_R\int_{T_1}^{T_2}(\Lambda_u+X_u+Y_u)du\right)\mid\mathcal{G}_t\right]\right)\\
&= \frac{1}{T_2-T_1}\left(\int_{T_1}^{T_2}\Lambda_u du+\bar{\alpha}(t,T_1,T_2)X_t+\bar{\beta}(t,T_1,T_2)Y_t\right)\\
&\quad+\frac{1}{\gamma_R}\frac{1}{T_2-T_1}\int_t^{T_2}\phi(\gamma_R\bar{\beta}(s,T_1,T_2))ds\\
&\quad+I_{\mathcal{G}}(t,T_1,T_2;T_\Upsilon)+\frac{\gamma_R}{2}\frac{1}{T_2-T_1}\int_t^{T_2}\sigma^2\breve{\alpha}^2(t,T_1,T_2;T_\Upsilon)ds
\end{aligned}
$$

where the solution of the last term is again given by equation Equation 7.31.

Now, the additional variance terms of the two preceding propositions need further investigation.

7.3.3. The Additional Variance Term

In this section we want to examine whether the complicated result derived in the preceding sections is plausible and no mistakes were made. To begin with, examining the individual expressions of Equation 7.31, it is quite easy to see that the variance is positive. In order to compare to the indifference price under the historical filtration we need the corresponding expression for the variance. Starting with the integral form of Benth et al. (2008a, Proposition 2.2) this is easily calculated as

$$
\begin{aligned}
\int_t^{T_2}\sigma^2\bar{\alpha}^2(s,T_1,T_2)ds &= \int_t^{T_1}\sigma^2(-\tfrac{1}{\alpha}(e^{-\alpha(T_2-s)}-e^{-\alpha(T_1-s)})^2ds\\
&\quad+\int_{T_1}^{T_2}\sigma^2(-\tfrac{1}{\alpha}(e^{-\alpha(T_2-s)}-1))^2ds
\end{aligned}
\tag{7.32}
$$

and hence the variance is given by

$$
\begin{aligned}
&Var\left(\int_{T_1}^{T_2}\int_t^u\sigma e^{-\alpha(u-s)}dW_s du\mid\mathcal{F}_t\right)\\
&\quad=\frac{\sigma^2}{\alpha^3}\left(\alpha(T_2-T_1)+\frac{1}{2}\left(e^{-2\alpha(T_2-T_1)}-e^{-2\alpha(T_2-t)}-e^{-2\alpha(T_1-t)}\right)\right)\\
&\quad+\frac{\sigma^2}{\alpha^3}\left(-1-e^{-\alpha(T_2+T_1)}\left(e^{2\alpha T_1}-e^{2\alpha t}\right)+2e^{-\alpha(T_2-T_1)}\right)
\end{aligned}
\tag{7.33}
$$

Now, we would like the variance under the market filtration to converge to the usual variance of Equation 7.33 for the case that T_Υ is located very far in the future.

Lemma 7.3.1. Upper limit of the new variance term. *The variance term of Equation 7.31 converges to the one of Benth et al. (2008a) as $T_\Upsilon\to\infty$.*

Proof. Applying Itō's isometry and the dominated convergence theorem to Equation 7.23 yields

$$\begin{aligned}
&\lim_{T_\Upsilon\to\infty}\int_t^{T_2}\left(\int_{s\vee T_1}^{T_2}\sigma e^{-\alpha(u-s)}C(s,u)du\right)^2 ds\\
&\qquad=\int_t^{T_2}\left(\int_{s\vee T_1}^{T_2}\sigma e^{-\alpha(u-s)}\lim_{T_\Upsilon\to\infty}C(s,u)du\right)^2 ds\\
&\qquad=\int_t^{T_2}\left(\int_{s\vee T_1}^{T_2}\sigma e^{-\alpha(u-s)}\lim_{T_\Upsilon\to\infty}\frac{e^{2\alpha T_\Upsilon}-e^{2\alpha u}}{e^{2\alpha T_\Upsilon}-e^{2\alpha s}}du\right)^2 ds\\
&\qquad=\int_t^{T_2}\left(\int_{s\vee T_1}^{T_2}\sigma e^{-\alpha(u-s)}du\right)^2 ds
\end{aligned}$$

which is exactly Equation 7.33 and thus the end of the proof. □

It is more difficult to understand what happens when T_Υ approaches T_2. This situation has been discussed in the literature and already in Pikovsky and Karatzas (1996) the authors realise a crucial point: if the investment period ends at or after the point in time for which future perfect information is available the expected utility of the insider is unlimited (in case there are no further restrictions on liquidity). Still, the product we are considering is of an averaging type (with a delivery from T_1 to T_2) and intuitively we would not expect an exploding utility in this situation. This in turn means that the variance need not be infinite.

Lemma 7.3.2. Lower limit of the new variance term. *The variance term of Equation 7.31 is finite for $T_\Upsilon \to T_2$. Furthermore, its value is given by*

$$\begin{aligned}
&\lim_{T_\Upsilon\to T_2} Var\left(\int_{T_1}^{T_2}\int_t^u \sigma e^{-\alpha(u-s)}dW_s du|\mathcal{F}_t\right)\\
&=\frac{\sigma^2}{\alpha^3}\left(\alpha(T_2-T_1)+\frac{1}{2}(e^{-\alpha T_1}(e^{2\alpha T_1}-e^{2\alpha T_2}))^2\left(\frac{1}{e^{2\alpha T_2}-e^{2\alpha T_1}}-\frac{1}{e^{2\alpha T_2}-e^{2\alpha t}}\right)\right)
\end{aligned}$$

Proof. Considering the first line of Equation 7.31 it is easy to see that this is finite when T_Υ approaches T_2. The same is true for the third line for which taking limits yields a zero contribution to the variance. For the second line things become slightly more complicated as the fraction inside the logarithm has a zero denominator when taking limits. Still, we can rearrange terms in order to apply L'Hospital's rule to the crucial expressions:

$$\begin{aligned}
&\lim_{T_\Upsilon\to T_2}\left(e^{-\alpha(T_\Upsilon-T_2)}-e^{-\alpha(T_2-T_\Upsilon)}\right)\log\left(\frac{(e^{\alpha T_\Upsilon}+e^{\alpha T_2})(e^{\alpha T_\Upsilon}-e^{\alpha T_1})}{(e^{\alpha T_\Upsilon}-e^{\alpha T_2})(e^{\alpha T_\Upsilon}+e^{\alpha T_1})}\right)\\
&=\lim_{T_\Upsilon\to T_2}\frac{\log(e^{\alpha T_\Upsilon}+e^{\alpha T_2})+\log(e^{\alpha T_\Upsilon}-e^{\alpha T_1})-\log(e^{\alpha T_\Upsilon}-e^{\alpha T_2})-\log(e^{\alpha T_\Upsilon}+e^{\alpha T_1})}{\left(e^{-\alpha(T_\Upsilon-T_2)}-e^{-\alpha(T_2-T_\Upsilon)}\right)^{-1}}
\end{aligned}$$

We can ignore all terms of the numerator but the (third) one including a difference in terms of T_Υ and T_2. Furthermore we can ease notation (and avoid multiple

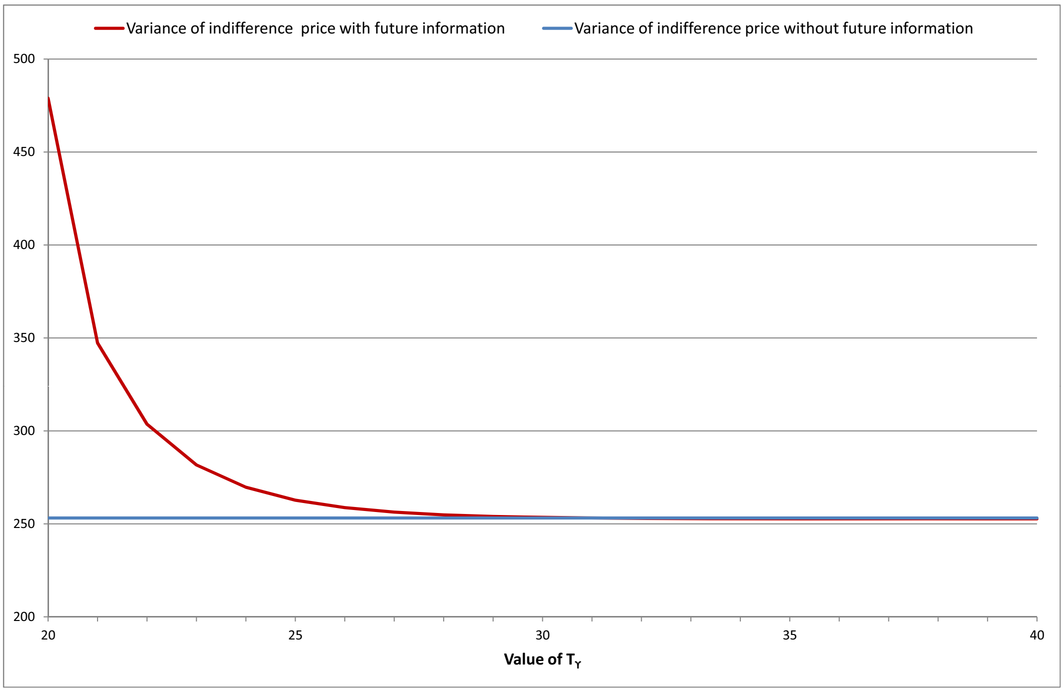


Figure 7.3.1.: Indifference prices: Variances of forward prices. Stylised example. Toy parameters chosen as follows: $t = 0, T_1 = 10, T_2 = 20, \alpha = 0.25, \sigma = 5$.

applications of L'Hospital later) by substituting $x = e^{\alpha T_\Upsilon}$ and $y = e^{\alpha T_2}$. Thus, we need to calculate

$$\lim_{x\to y} \frac{-\log(x-y)}{\frac{1}{\frac{y}{x}-\frac{x}{y}}} \overset{L'H}{=} \lim_{x\to y} \frac{-\frac{1}{x-y}}{\frac{1}{(\frac{y}{x}-\frac{x}{y})^2}(-y\frac{1}{x^2}-\frac{1}{y})} = \lim_{x\to y} \frac{\frac{1}{y}\frac{1}{x-y}(\frac{y}{x}-\frac{x}{y})^2}{\frac{1}{x^2}+\frac{1}{y^2}}$$
$$= \frac{1}{\frac{1}{y^2}+\frac{1}{y^2}} \lim_{x\to y} \frac{(\frac{y}{x}-\frac{x}{y})^2}{xy-y^2} \overset{L'H}{=} \frac{y^2}{2} \lim_{x\to y} \frac{2(\frac{y}{x}-\frac{x}{y})(y\log(x)-\frac{1}{y})}{y} = 0$$

Hence, all parts of the variance are finite when taking limits. The resulting expression for the whole variance is then straightforwardly calculated. □

The convergence results (in both directions) of the variance term under the market filtration are illustrated against the variance under the historical filtration in Figure 7.3.1. They do confirm the analytical investigation of this chapter: a finite but larger value for $T_\Upsilon = T_2$ (which is 20 here) and a converging variance for T_Υ moving away from the delivery period.

7.4. Contribution and Discussion

The contribution of this chapter is twofold. Firstly, by calculating expressions for indifference prices of electricity forwards with our extended spot-forward relationship we allow for a variety of interesting questions in the framework of Benth et al.

(2008a). For example, one might ask how market power changes when the producer, say, uses the new indifference price and the retailer the old one. Another aspect could be to explore the shift in prices and market power when both agents have access to additional information. We have already commenced following this path of research when interpreting the resulting additional variance term in Section 7.3.3.

The second (mathematical) contribution of this chapter is that it is, to the best of our knowledge, the first time that a different combination other than a log-normal underlying and a log-utility are applied to an enlargement of filtration framework. There is a large number of papers on modelling insider trading all of which use this setup when calculating the insider's utility. Clearly, the reason for this is that only with this combination the logarithm of the utility and the exponential of the underlying cancel making it easy to straightforwardly apply results from the theory of enlargement of filtration. In this chapter we show that closed-form solutions are available for other combinations of utility function and underlying and in our case we consider an arithmetic underlying and an exponential utility. This is achieved by assuming the future information to be perfect on the one hand while on the other hand viewing enlargement of filtration from a fundamentally different point of view. Treating the decomposition as a stochastic differential equation we can deduce distributional properties even of functionals of the corresponding expressions.

A lot of future research could be motivated by the calculations in this chapter. It would be interesting to see empirical studies along the lines of those in Benth et al. (2008a) utilising the new indifference prices. Furthermore, the new approach to enlargement of filtration should be given some thought, in particular in a review of the insider trading literature. This actually is the subject of another section of the working paper mentioned in Section 1.4.2 of the introduction.

Chapter 8.

Existence of the Information Premium: A Statistical Test

8.1. Literature Overview and Summary

So far, in this thesis, we have seen an analytical and theoretical treatment of the concept of the information premium on electricity markets. We have thus far illustrated our calculations making use of stylised examples. In this chapter though, we will finally present a statistical test for the existence of the information premium as well as a thorough empirical study. The basis of this chapter is the publication Benth, Biegler-König, and Kiesel (2013a).

Dealing with the information premium empirically turns out to be non-trivial, the main reason being that the premium is not measurable with respect to the historical filtration. The orthogonality of the information premium with respect to the space spanned by the historical filtration was shown in the key Lemma 4.2.2. Normally, to identify objects like the risk premium the usual textbook approach in Financial Mathematics is to (directly or impliedly) change measure. This road is followed in a myriad of papers: Let us only mention classical empirical studies such as Macbeth and Merville (1979) for Black-Scholes stocks and options, Schwartz (1997) for various commodities or Benth et al. (2008a) for electricity (cf. Chapter 7).

In this chapter we will propose another method that will involve regressions and Hilbert space representations. Our statistical test will also provide a time series for the information premium whose features will match our economic intuition in size and shape. Furthermore, let us comment on the generality of our methodology: Financially, it is applicable for testing for different information sets in any market. Mathematically, to the best of our knowledge, no other method to empirically test for non-measurability has been brought forward in the literature yet.

This chapter will be structured as follows: in Section 8.2 we will propose the statistical test, relying on theory from Section 2.3. We will present an innovative method to empirically change measure in Section 8.3. We will then apply the test in Section 8.4 conducting case studies for both market scenarios mentioned before in Section 1.2. We will add qualitative discussions to illustrate the economic background. We will conclude that the introduction of the second phase of CO_2 certificates as well as the German Atom Moratorium resulted in huge information premia.

The first part of Section 8.5 will then take related underlyings such as fuels (i.e. oil and gas) but also stock indices into consideration. The reason for this additional analysis will be to examine whether the information premia identified for both scenarios might possibly be induced by some other event. Section 8.5.2 will discuss robustness issues of the statistical test by making use of simulated scenarios.

Finally, Section 8.6 will summarise and conclude.

8.2. A Test for the Information Premium

The goal of this section is to design a general statistical test to show the existence of the information premium empirically. As mentioned before, Lemma 4.2.2 tells us that the information premium is orthogonal to the space spanned by $\mathcal{F}_t$, independent of the underlying measure and thus making a classical change of measure approach impossible. Instead, we want to isolate an estimator for the information premium, i.e. for the object defined as

$$I_{\mathcal{G}}^{\mathbb{Q}}(t, T_1, T_2) = F_{\mathcal{G}}^{\mathbb{Q}}(t, T_1, T_2) - F_{\mathcal{F}}^{\mathbb{Q}}(t, T_1, T_2) \tag{8.1}$$

in Definition 5.3.1.

The first step of our method is to calibrate our two-factor arithmetic spot price model according to the description in Section 3.7. Then, remembering the modified spot forward relationship we proposed earlier (cf. Proposition 4.2.2) we also believe that a measure change is taking place on the market. Consequently, in a second step, we will choose a pricing measure $\mathbb{Q}$ by identifying measure change parameters that minimise the distance between observed and calculated forward prices. We will provide details about this procedure later. This will allow us to calculate forward prices under measure $\mathbb{Q}$ and (known) filtration $\mathcal{F}_t$:

$$F_{\mathcal{F}}^{\mathbb{Q}}(t, T_1, T_2) = \frac{1}{T_2 - T_1} \mathbb{E}^{\mathbb{Q}} \left[\int_{T_1}^{T_2} S_u du \mid \mathcal{F}_t \right] \tag{8.2}$$

We refer the reader to Section 3.4 for details. Furthermore, let us also remark that the change of measure will only normalise the object analysed in the following, once again because Lemma 4.2.2 is valid for all equivalent measures.

Without any assumptions about the additional information available to market agents (one such example would be the threshold information discussed in Example 4.2.1) we cannot make concrete statements about the structure of the market filtration $\mathcal{G}_t$. In particular, this means we cannot simulate or calculate expectations (i.e. the information premium) with respect to this filtration.

Given the framework of Chapter 4 we generalise our method under the following assumption:

Assumption 8.2.1. Construction of observed market prices. With Notation 4.2.1 we will assume that

$$\hat{F}(t, T_1, T_2) \overset{!}{=} F_{\mathcal{G}}^{\mathbb{Q}}(t, T_1, T_2)$$

meaning that observed prices are calculated by market traders according to the new spot-forward relationship (cf. Proposition 4.2.2).

We can now combine Equation 8.2 and Assumption 8.2.1 to rewrite Equation 8.1 as an empirical version

$$\hat{I}_{\mathcal{G}}^{\mathbb{Q}}(t, T_1, T_2) = \hat{F}(t, T_1, T_2) - F_{\mathcal{F}}^{\mathbb{Q}}(t, T_1, T_2) \tag{8.3}$$

The object $\hat{I}_{\mathcal{G}}^{\mathbb{Q}}(t, T_1, T_2)$ will be our best estimate for the information premium. We will need to verify that it satisfies the properties we have identified. We can only claim the premium exists in case $\hat{I}_{\mathcal{G}}^{\mathbb{Q}}(t, T_1, T_2)$ is non-zero. Furthermore, we know that the premium is not $\mathcal{F}_t$-measurable, i.e. it needs to satisfy the orthogonality property. Summarising, this translates to:

- $\hat{I}_{\mathcal{G}}^{\mathbb{Q}}(t, T_1, T_2)$ is significantly not white noise
- $\hat{I}_{\mathcal{G}}^{\mathbb{Q}}(t, T_1, T_2)$ is orthogonal to the space $L^2(\mathcal{F}_t, \mathbb{Q})$ spanned by the historical filtration, or equivalently $\mathbb{E}[\hat{I}_{\mathcal{G}}^{\mathbb{Q}}(t, T_1, T_2)|\mathcal{F}_t] = 0$

Testing $\hat{I}_{\mathcal{G}}^{\mathbb{Q}}(t, T_1, T_2)$ for white noise is fairly standard. A simple graphical method is to analyse the graph of $\hat{I}_{\mathcal{G}}^{\mathbb{Q}}(t, T_1, T_2)$ as well as the graph of its auto-correlation function. We can also apply standard auto-correlation-based tests like the *Ljung-Box* test (cf. Vogelvang (2005, page 121) for details). We will do both in the case studies of Section 8.4.

The second property is much more difficult to handle. As mentioned in Section 8.1 we propose a method based on Hilbert space representations and regressions. We already summarised some of the most important theoretical concepts in Section 2.3.

We want to show empirically that the conditional expectation under the historical filtration of our estimator satisfies

$$\mathbb{E}\left[\hat{I}_{\mathcal{G}}^{\mathbb{Q}}(t, T_1, T_2) \mid \mathcal{F}_t\right] = 0 \tag{8.4}$$

We will assume that we are in a Hilbert space setting and that all our objects are in either $L^2(\mathcal{G}_t, \mathbb{Q})$ or $L^2(\mathcal{F}_t, \mathbb{Q})$. Now, part (1) of Theorem 2.3.2 tells us that the conditional expectation operator is a contraction of the underlying Hilbert space. Hence, we know that

$$\mathbb{E}\left[\hat{I}_{\mathcal{G}}^{\mathbb{Q}}(t, T_1, T_2) \mid \mathcal{F}_t\right] \in L^2(\mathcal{F}_t, \mathbb{Q})$$

Furthermore, we have established in part (2) of Theorem 2.3.2 that there exists a functional form $F : (\Omega, \mathcal{F}_t) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ of the conditional expectation. Clearly, the historical filtration is given by $\mathcal{F}_t = \sigma(S_u; 0 \leq u \leq t)$ and the spot is Markovian and thus the functional takes the form

$$\mathbb{E}\left[\hat{I}_{\mathcal{G}}^{\mathbb{Q}}(t, T_1, T_2) \mid \mathcal{F}_t\right](\omega) = F(S_t(\omega))$$

But now, by the fundamental result on Hilbert spaces (cf. Theorem 2.3.1), we can find a linear combination of elements of an orthonormal system $\{\phi_1(S_t), \phi_2(S_t), \ldots\}$ such that

$$F(S_t) = \sum_{i=1}^{\infty} c_i \phi_i(S_t)$$

In practice, the values of the coefficients c_i can now (for a sufficiently large number of basis elements) be identified by means of a least squares regression from a sample of values of S_t onto a sample of $F(S_t)$. This is the framework of the famous

Least Squares Monte Carlo method that we will briefly discuss and compare to our methodology in Section 8.2.1.

One of the main differences is that, as mentioned above and in contrast to Least Squares Monte Carlo, we cannot simulate numerical values for the functional form $F(S_t)$ as we have not assumed any knowledge of the filtration $\mathcal{G}_t$. The market data we have access to, though, consists of daily spot prices as well as daily forward prices. We will therefore consider specific forward contracts and their "lifetime" (i.e. the time they are traded and priced on the market) and let

$$t_0 < \ldots < t_k < \ldots < t_n = T_2$$

be the discrete grid of (daily) time points of that time span. Then we have S_{t_k} and can calculate the value of the estimator $\hat{I}^{\mathbb{Q}}_{\mathcal{G}}(t_k, T_1, T_2)$ for every day t_k. To make these two time series comparable and to identify coefficients by means of regression the elements of the time series need to have the same distribution. Hence, we will consider their stationary increments. We will test for stationarity (with *Dickey-Fuller*, cf. Davidson and MacKinnon (1993, Section 20.2) for details) and it will turn out in Section 8.4 that for both time series S_t and $\hat{I}^{\mathbb{Q}}_{\mathcal{G}}(t, T_1, T_2)$, the time series of first differences, denoted by $\triangle S_t$ and $\triangle \hat{I}^{\mathbb{Q}}_{\mathcal{G}}(t, T_1, T_2)$, are stationary, i.e.

$$\triangle S_k \sim \triangle S_l \ , \ \triangle \hat{I}^{\mathbb{Q}}_{\mathcal{G}}(t_k, T_1, T_2) \sim \triangle \hat{I}^{\mathbb{Q}}_{\mathcal{G}}(t_l, T_1, T_2) \ \forall k, l$$

For now, we will just think of the notation $\triangle$ as denoting the stationary increments.

We seek to find the functional form F such that

$$\mathbb{E}\left[\triangle \hat{I}^{\mathbb{Q}}_{\mathcal{G}}(t, T_1, T_2) \mid \mathcal{F}_t\right](\omega) = F(\triangle S_t(\omega))$$

In other words, we need to find the coefficients c_i that satisfy

$$\triangle \hat{I}^{\mathbb{Q}}_{\mathcal{G}}(t, T_1, T_2) = \sum_{i=1}^{\infty} c_i \phi_i(\triangle S_t)$$

For a suitable number N of basis elements the corresponding regression formula is

$$\triangle \hat{I}^{\mathbb{Q}}_{\mathcal{G}}(t, T_1, T_2) = \sum_{i=1}^{N} c_i \phi_i(\triangle S_t) + \triangle \epsilon(t) \tag{8.5}$$

This regression will finally yield an estimate of the desired functional form of the conditional expectation. We want to test how much of the variation in $\hat{I}^{\mathbb{Q}}_{\mathcal{G}}(t, T_1, T_2)$ can be explained by S_t (and thus by filtration $\mathcal{F}_t$). We will thus conclude non-measurability if the regression yields little explanatory power and all regression coefficients are insignificant. $\mathbb{E}[\hat{I}^{\mathbb{Q}}_{\mathcal{G}}(t, T_1, T_2) \mid \mathcal{F}_t] = 0$ will follow from this. We will use R^2, the F-statistic and, more importantly, individual t-statistics as a measure of significance in the empirical analysis.

Summarising, in this section, we propose the following statistical test for the existence of the information premium:

Algorithm 8.2.1. Test for the existence of an information premium. Given a time series of spot price data as well as a matrix of forward prices we go through the following steps:

1. Calibrate the spot model to observed spot data $\hat{S}_t$
2. Calculate forward prices $F^{\mathbb{P}}_{\mathcal{F}}(t, T_1, T_2)$
3. Conduct a measure change to find the risk-neutral $\mathbb{Q}$ that minimises the difference between $F^{\mathbb{Q}}_{\mathcal{F}}(t, T_1, T_2)$ and observed forward prices $\hat{F}(t, T_1, T_2)$
4. Calculate forward prices $F^{\mathbb{Q}}_{\mathcal{F}}(t, T_1, T_2)$
5. Choose a certain forward to be examined and assemble its time series from birth until death
6. Calculate the residual term $\hat{I}^{\mathbb{Q}}_{\mathcal{G}}(t, T_1, T_2)$ for all days of the lifetime of the forward
7. Check that $\hat{I}^{\mathbb{Q}}_{\mathcal{G}}(t, T_1, T_2)$ is not white noise (i.e. the information premium exists)
8. Test $\hat{I}^{\mathbb{Q}}_{\mathcal{G}}(t, T_1, T_2)$ for non-measurability
   - Take (stationary) first differences of time series $\hat{I}^{\mathbb{Q}}_{\mathcal{G}}(t, T_1, T_2)$ and $\hat{S}_t$
   - Regress the two time series and check significance (F- and t-statistics)

Our test confirms the existence of an information premium if R^2 is small and the F-statistic and t-statistics suggest zero-coefficients. If so, $\hat{I}^{\mathbb{Q}}_{\mathcal{G}}(t, T_1, T_2)$ is an estimator for the information premium.

As announced above, we will now briefly compare the test as summarised in Algorithm 8.2.1 to the classical Least Squares Monte Carlo method.

8.2.1. Excursus: The Least Squares Monte Carlo Method

The idea to combine regression and simulation in order to calculate the value of conditional expectations was introduced in the context of the valuation of American options in Longstaff and Schwartz (2001), Tsitsiklis and Van Roy (2001) as well as Carriere (1996). The former of these papers is by far the most widely known one. The idea of Longstaff and Schwartz (2001) is as follows: we want to estimate the expected payoff from continuation (i.e. a conditional expectation) at some intermediate time point. We therefore simulate the underlying over a grid of time points. The payoff at expiry is easily calculated. Then, we move back in time and run a regression from current values of simulated paths to the calculated prices of each path at the next time point. This regression will provide a functional form for the conditional expectation and we can compare to the value of exercising immediate. Repeating this procedure recursively will finally yield an estimator for the fair price of the American option.

Generally, the so-called Least Squares Monte Carlo method has had a great impact and many other applications have been discussed in the literature. Convergence

Method	Classical LSMC	New method
Time	fixed t	$t_k \in \{t_0, \ldots, t_n\}$
Regressor	simulated X_t	stationary $\triangle X_{t_k}\ \forall k$
Regressand	simulated $F(X_{t+1})$	stationary $\triangle F(t_k)\ \forall k$
Goal	Value of cond. expectation	Quality of regression

Table 8.2.1.: Comparison between methods. The classical LSMC method and our new method

results for the algorithm are provided in Stentoft (2004) and Clément et al. (2002). Longstaff and Schwartz (2001) provide a discussion on which basis to choose. It turns out that a clever choice of basis for the functional representation of the conditional expectation can greatly improve convergence rates. In the field of energy finance one example of an application is Kiesel et al. (2010) where the authors use the method to calculate prices of complicated swing options on electricity.

Table 8.2.1 summarises the differences between the classical method and the one we have proposed in the previous section.

There are two main differences. Firstly, as mentioned earlier, regressor and regressand need to be simulated in the classical method. This is impossible in our case as we have only observed prices to work with and cannot simulate $\hat{I}_{\mathcal{G}}^{\mathbb{Q}}(t, T_1, T_2)$ without imposing a structure onto the market filtration and thus removing generality. Secondly, both methods have different goals. Whereas the classical approach seeks for a concrete price we are more interested in the quality of the regression. This is also why for our method the choice of basis is of lesser importance.

8.2.2. Further Remarks on the Test

Here are some supplementing remarks and answers to possible criticism considering the method we have proposed.

We have thus far left unanswered which Hilbert basis to use when applying our method. In Section 8.4 we will make use of only the simple polynomial basis. This is given by the monomials $\{X_{t_k}^0, X_{t_k}^1, X_{t_k}^2, \ldots\}$. Even though some research has been conducted as to the benefits of using different bases in the case of the classical Least Squares Monte Carlo method (see above) this choice is more or less irrelevant in our case. After all, we are only interested in the significance properties and not the exact result or the speed of convergence of the algorithm. There is a bijection between different complete orthonormal systems and by making use of a sufficiently large number of basis elements we can constrain ourselves to the choice of the polynomial basis. We will mostly use ten basis elements in the next section. We claim that this is more than enough, especially when taking into consideration the length of the datasets analysed. We will indeed find that increasing the number of basis elements will not alter results.

Furthermore, one could postulate that our findings might be due to bad fitting of the spot price or, more generally, a bad or insufficient spot model. Still, we have seen that we use mostly observed data in our method. The only objects depending on the spot model are the forward prices calculated under the pricing measure and the historical filtration. Still, these will turn out to be practically piecewise constant and thus will not contribute to the variation of prices and consequently to regression results. This fact is relatively independent of the choice of spot model for two reasons. Firstly, the speeds of mean-reversion usually observed on electricity markets imply a half-life (calculated as $\frac{\ln 2}{\alpha}$ for a speed of mean-reversion α) of only a handful of days. Hence, the influence of the current spot price diminishes quickly. Secondly, forwards with delivery period are calculated as integrals over a time period. We will examine forwards with a delivery period of one month. This smoothes out any possible variations that may arise. Summarising, we claim that our method works well, even for an oversimplifying spot model.

8.3. Empirical Measure Change and Structure of Forward Prices

In this section we will present the way in which we will change from measure $\mathbb{P}$ to $\mathbb{Q}$ in more detail. Furthermore, we will have a look at a sample of EEX data in order to discuss results and the relationship between the different observed and calculated objects.

8.3.1. Futures Data and Measure Change

The first difficulty one encounters when changing measure is how to read observed futures prices (for a discussion on forwards and futures we refer to Section 1.1.2). Data providers such as Bloomberg, Reuters and the EEX provide daily data organised in columns each representing one of the different classes of forwards traded on that particular day. These classes have delivery in a certain number of months, or quarters, or years. This means that a rolling effect occurs, i.e. that if one wants to track a certain contract through its lifetime one has to read the data table diagonally in a top-right to bottom-left direction. For example, the January 2012 forward is in the six-months column for August 2011 and in the two-months column for December 2011. Hence, there are essentially three ways to examine forward prices. We can consider:

- the forward contract maturing in a certain number of months for all days (i.e. t and $[T_1, T_2]$ moving through the time series). This corresponds to reading a single column of the data table.
- the forward contract maturing in a specified month (i.e. t moving through the lifetime of the forward, T_1, T_2 fixed). An example would be Figure 1.2.4. This corresponds to reading the table diagonally.
- all forward contracts traded on a specified day (i.e. with fixed t). An example would be Figure 1.2.2. This corresponds to reading a single row of the table.

In Section 3.4 we have seen the formulae for the prices of forwards with delivery period and under both the real-world and the pricing measure. In this (empirical) chapter we will work with an easy framework and only consider a Girsanov measure change with constant parameters. A generalisation to non-constant Radon-Nikodým derivatives or the Esscher transform can easily be implemented.

As mentioned before, empirically, we choose Girsanov parameters in such a way as to minimise the difference between the observed forward prices and the calculated $\mathbb{Q}$-prices. The way this is done is on a global scale (i.e. over the length of the whole dataset under consideration) for each class of forwards. This means that constant distance-minimising parameters are identified (by means of least squares) for each time until delivery, for example the three-month-forward or the four-month-forward (this is the first approach as to reading the data table described above).

The reason for this approach is that we believe that market participants adjust risk for (global) classes of forward contracts. Generally, traders will price a two-month (time-to-delivery) forward differently than a six-month (time-to-delivery) forward. Still, all two-month forward contracts will be similarly risk-adjusted by market traders. In particular, this is in line with the risk premium literature discussed in Section 4.1 and especially with Benth et al. (2008a) (cf. Chapter 7). We remember that in this paper the authors find positive risk premia for short term deliveries and negative ones for long term delivery periods. This (non-orthogonal) effect will be captured by our way of changing measure. We will discuss this issue for both case studies in Section 8.4.1 and Section 8.4.2. Generally, the idea we propose is quite powerful, while at the same time it provides a lot of flexibility.

8.3.2. The Structure of Different Forward Prices

To provide an overview of the relation between different observed and calculated objects we will now briefly look at the EEX dataset ranging from 06/10/2003 until 26/05/2006. The spot price of this range is illustrated in Figure 8.3.1. Calibration results are provided in Section B.1.

We will now consider forwards with different and fixed time until delivery period. This will be done by comparing observed and realised prices for a forward delivering in a certain number of months or quarters with calculated prices for the same delivery period. The following three figures will show all four of these objects. We remark that the realised price is calculated as the forward looking arithmetic mean of spot prices during the delivery period under consideration. As we have a fixed time-horizon, this time series will be shorter than the others.

Figure 8.3.2 shows prices for the one-month forward, i.e. the current month. All graphs, especially those two calculated are very close to each other. The most striking part of this figure happens during the winter 2004/2005 where at the beginning of each month the calculated prices very much overestimate the observed and realised prices. This is followed by a sharp decrease until the realised price is reached at the end of the month. The reason for this behaviour lies in Corollary 3.4.1: The price of the forward for the current month is the expected value of future days plus the arithmetic mean of that part of the month already in the past. Our spot model does not seem to fit this period very well, the estimation gradually being corrected by the arithmetic mean terms.

Figure 8.3.3 shows the corresponding picture for six months until delivery period. Here, we see that calculated forward prices are piecewise constant. The reason for

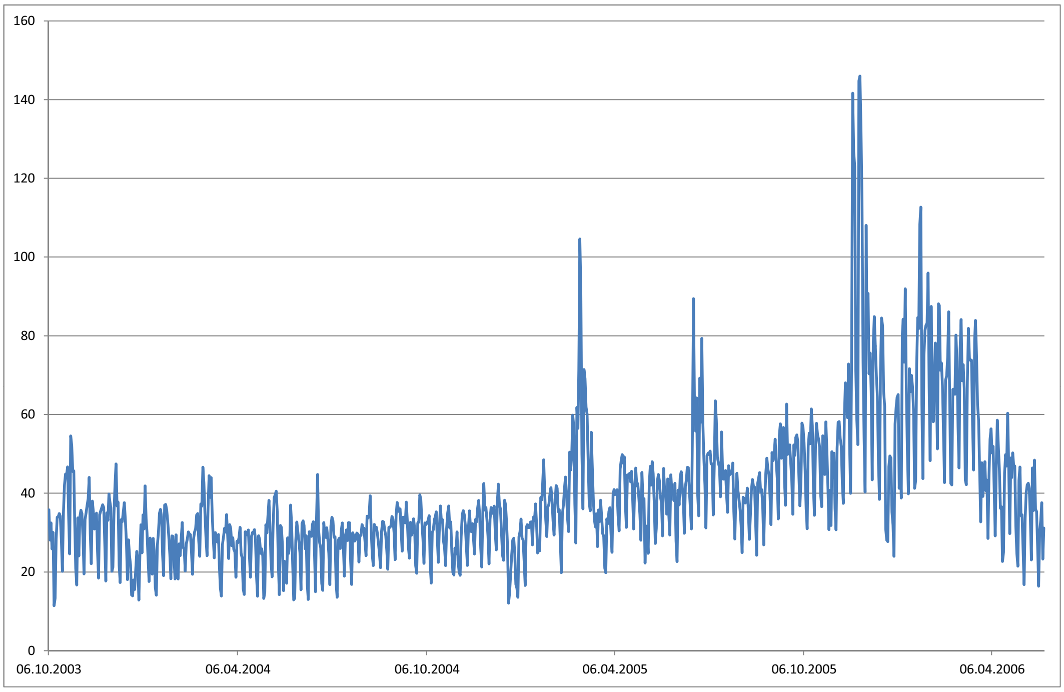


Figure 8.3.1.: EEX data from 10/03 until 05/06: Spot price. Observed from 06/10/2003 until 26/05/2006. In Euros.

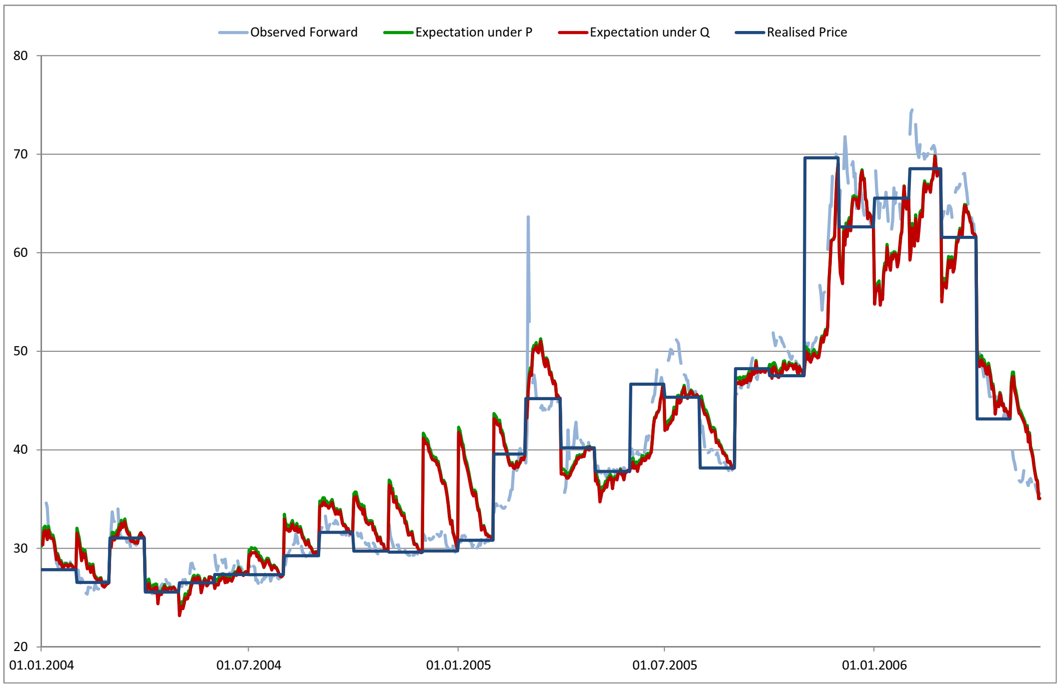


Figure 8.3.2.: EEX data from 10/03 until 05/06: One-month forward prices. Observed, realised and calculated under both measures. In Euros.

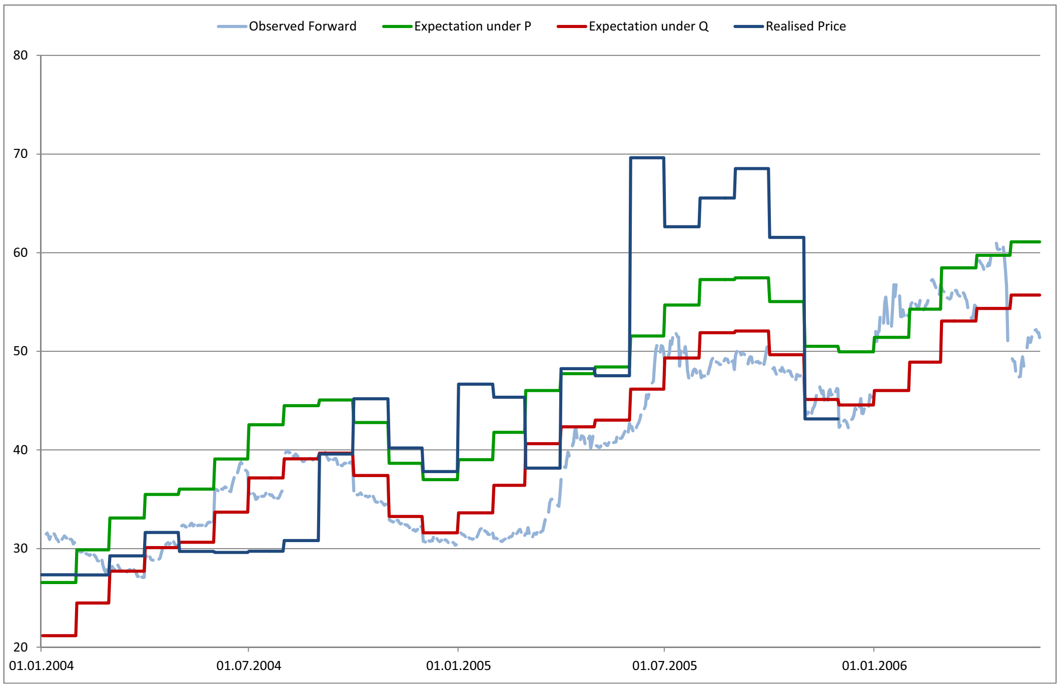


Figure 8.3.3.: EEX data from 10/03 until 05/06: Six-month forward prices. Observed, realised and calculated under both measures. In Euros.

this general phenomenon is the size of the fitted mean-reversion parameters which flattens out prices (i.e. the influence of current spot prices becomes negligible). Table B.1 provides the value of $\alpha = 0.31$ inducing a half-life of $\frac{\ln 2}{\alpha} = 2.24$ days. The difference between the two calculated time series is now much bigger (more than five Euros) and one can clearly see that the risk-neutral price is very close to observed prices. Another typical feature is that the real-world price is closer to the actual price realised later. The same fact can be observed in Figure 8.3.4 which shows prices with a delivery period of a quarter and a time to maturity of four quarters, i.e. one year. Again, the risk-neutral price is coupled to the observed price whereas the real-world price is closer to the realised price.

Summarising, this phenomenon grows in size with time to maturity. The further the delivery period the larger the effect of the measure change (i.e. the risk-adjustment) and the more extreme and obvious is the coupling between observed and risk-neutral prices on the one hand and realised and real-world prices on the other hand.

8.4. Empirical Studies

Finally, as we have gathered all the techniques necessary, we can now examine the two scenarios presented in the introduction of this thesis: the beginning of the second phase of the EUETS and the German Atom Moratorium. Our goal is to verify the existence of significant information premia during both of these market situations

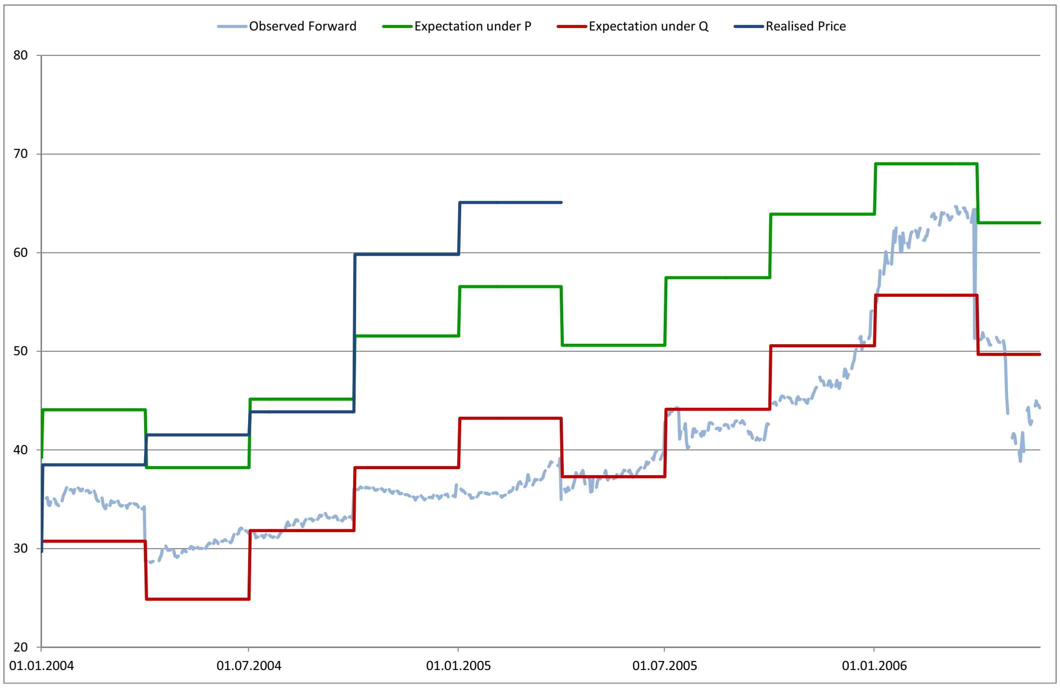


Figure 8.3.4.: EEX data from 10/03 until 05/06: Four-quarter forward prices. Observed, realised and calculated under both measures. In Euros.

as hypothesised in Section 1.2.

8.4.1. The Second Phase of the EUETS

The data set under consideration in this section will consist of EEX prices from 01/02/2007 until 30/10/2008, consisting of 639 days of data. The dataset was chosen as to include the crucial date 01/01/2008 (i.e. the beginning of the second phase of CO_2-certificates) as a midpoint.

The spot for this range of dates is illustrated in Figure 8.4.5. Already in this figure can we observe the introduction of the CO_2-certificates as December 2007 and January 2008 are very volatile and separate the data set into two parts. In 2007 spot prices are around 30 Euros with relatively little volatility and the prices in 2008 are at least around 50 to 60 Euros with a slightly higher volatility. Thus, prices exhibit not only the usual, slow positive trend but also a general shift upwards.

Details about the calibration of the spot model and some robustness and performance results are provided in Section B.2: Table B.4 provides parameter values and robustness, Table B.5 compares observed moments with simulated ones.

Change of measure parameters are given in Table 8.4.2. They are positive for the first three months and negative for more distant delivery periods. This is the same type of behaviour that we witnessed for the dataset of the last section (remember Table B.3). This confirms nicely the findings of Benth et al. (2008a): there is a positive risk premium for delivery periods in the near future due to hedging pressure of retailers. Also, for long times until delivery the risk premium induced by these

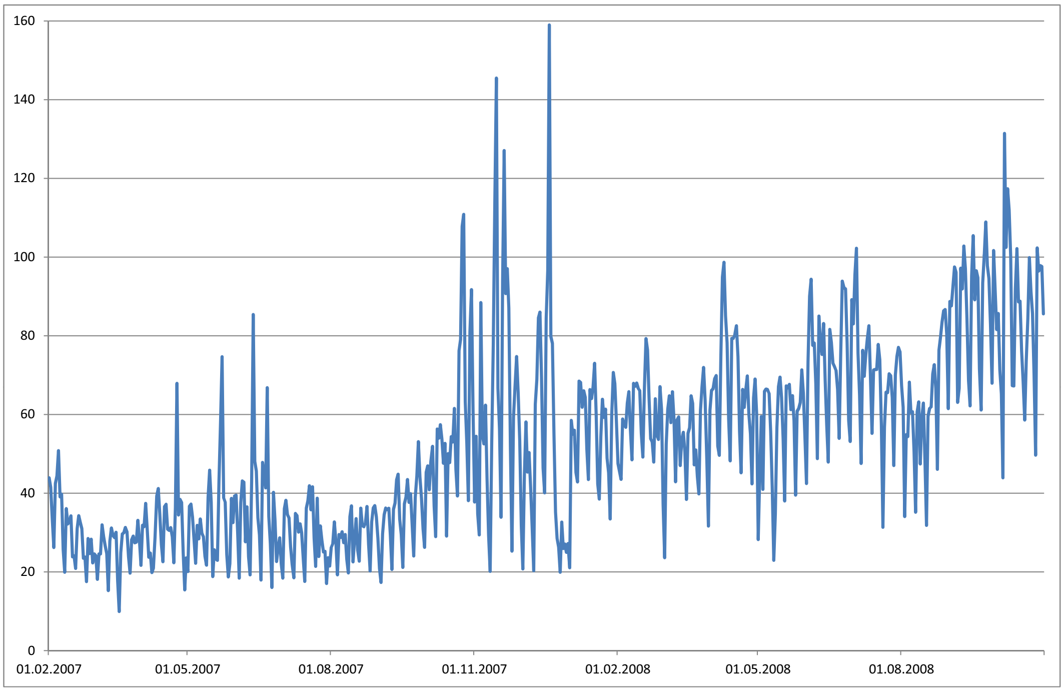


Figure 8.4.5.: EUETS case study: Spot price. EEX data from 01/02/2007 until 30/10/2008. In Euros.

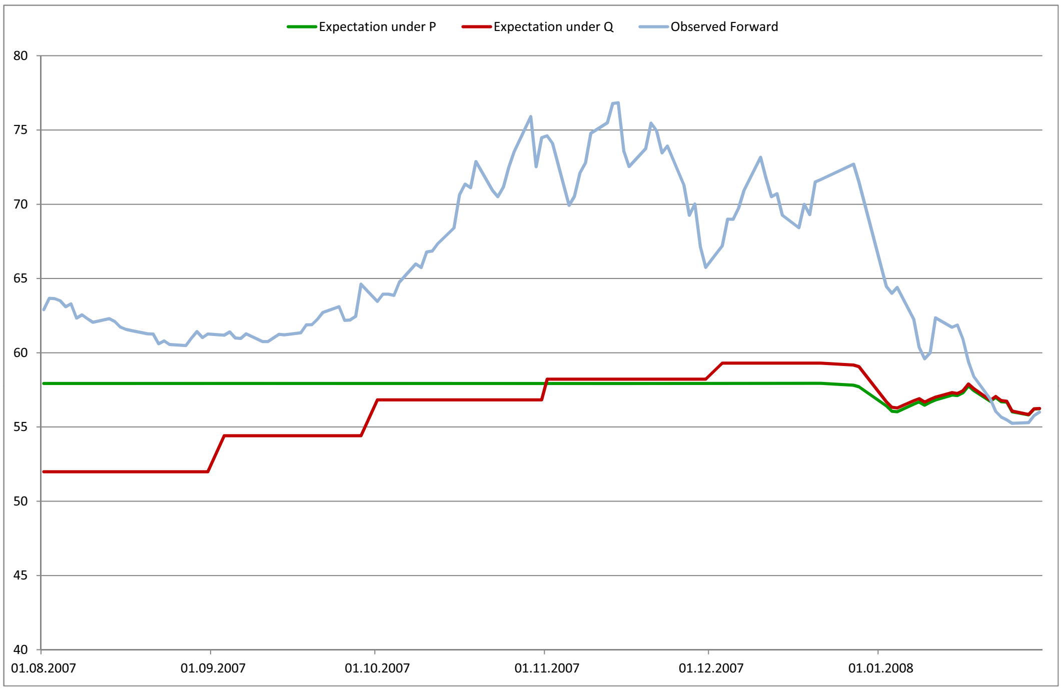


Figure 8.4.6.: EUETS case study: January 2008 prices. Observed and calculated forward prices with delivery in January 2008. In Euros.

Forward	1 m	2 m	3 m	4 m	5 m	6 m
θ_W	0.164	0.734	0.153	-0.593	-1.893	-3.199

Table 8.4.2.: EUETS case study: Girsanov parameters. For different forward classes.

parameters is negative because producers desire a constant and non-volatile stream of revenues.

Now, we assemble the different objects that we will need in order to conduct the test as described in Section 8.2. For the forward with delivery period in January 2008 these are shown in Figure 8.4.6. This forward was born (i.e. first traded) on 01/08/2007. Its lifetime ended on 31/01/2008. On the one hand, we can see the constant behaviour of the real-world forward for reasons mentioned in Section 8.2.2. On the other hand, we perceive the piece-wise constant behaviour of the forward under the pricing measure. This is due to our way of changing measure which is different for each phase of the life of the forward.

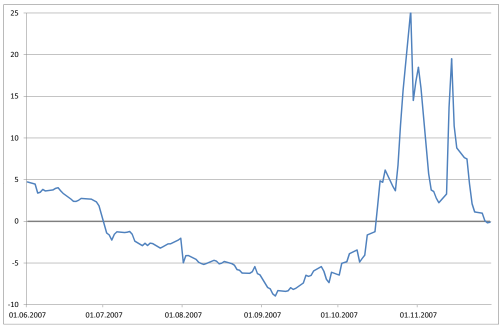
With these three time series we can now empirically calculate the traditional risk premium $R^{\mathbb{Q}}_{\mathcal{F}}(t, T_1, T_2)$ (by subtracting from observed prices the expected price under the real-world measure) as well as our estimator $\hat{I}^{\mathbb{Q}}_{\mathcal{G}}(t, T_1, T_2)$.

8.4.1.1. The Second Phase of the EUETS: Estimator

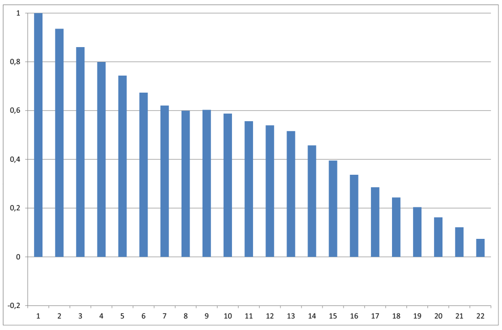
As mentioned before, we now change perspective by examining one special forward at any one time. The objects depicted in Figure 8.4.6 allow to calculate the estimator $\hat{I}^{\mathbb{Q}}_{\mathcal{G}}(t, T_1, T_2)$ according to Equation 8.1. In our case study, we consider contracts living before, after and during the introduction of the CO_2-certificates. We choose the contracts with delivery period in November 2007 ($t_0 = 01/06/2007$, $T_1 = 01/11/2007$ and $T_2 = 30/11/2007$), January 2008 ($t_0 = 01/08/2007$, $T_1 = 01/01/2008$ and $T_2 = 31/01/2008$), March 2008 and August 2008. The resulting estimators and corresponding auto-correlation functions are depicted in Figure 8.4.7.

The most interesting part of this figure is Subfigure 8.4.7c, i.e. the residual for the contract with delivery in January 2008. It is large and positive for almost the whole time span and has a larger volatility the closer time until delivery comes. Then, starting from 01/01/2008 it decreases to approach zero on 31/01/2008. This is exactly the type of behaviour one would have expected from the motivating example: a positive information premium which tends to zero after the spot itself finally reacts in real-time to the introduction of phase two emission certificates. The picture is very similar for the March 2008 forward (Subfigure 8.4.7e) where one again finds positive values for the months of 2007 followed by negative values for 2008. This can be interpreted presuming that the market expected a price increase due to CO_2-certificates which was overestimated. The graphs of the estimators for November 2007 and August 2008 show less regular behaviour and are of smaller size. Table 8.4.3 summarises important properties of the four estimators confirming our qualitative analysis.

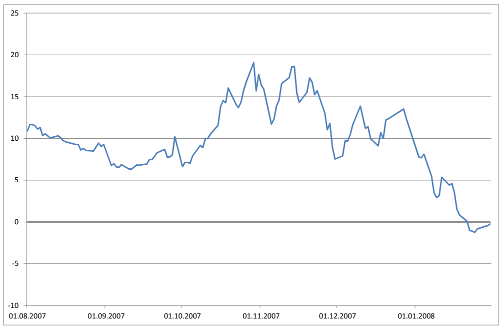
In order to verify that the estimators $\hat{I}^{\mathbb{Q}}_{\mathcal{G}}(t, T_1, T_2)$ are significantly non-zero we



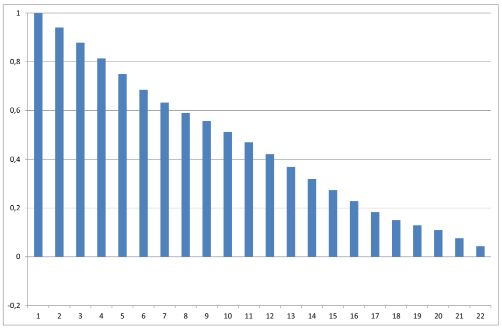
(a) November 2007: Information premium est.



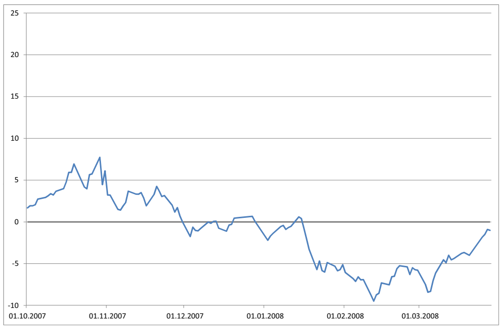
(b) November 2007: acf of estimator



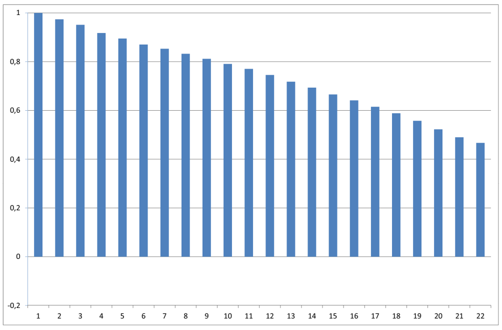
(c) January 2008: Information premium est.



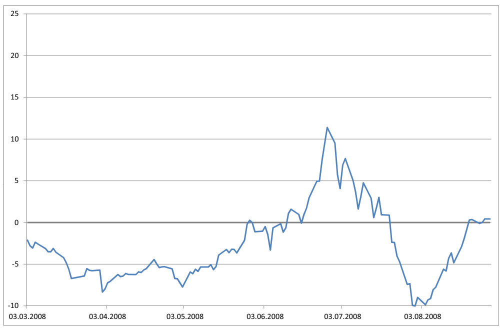
(d) January 2008: acf of estimator



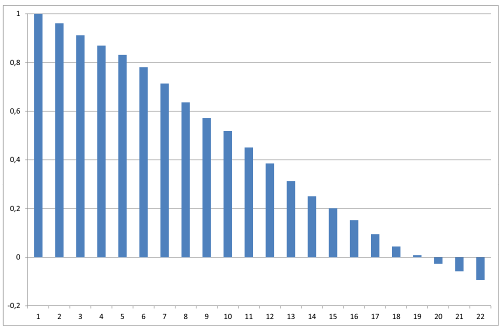
(e) March 2008: Information premium est.



(f) March 2008: acf of estimator



(g) August 2008: Information premium est.



(h) August 2008: acf of estimator

Figure 8.4.7.: EUETS case study: Estimators. And auto-correlation functions.

	Nov 07	Jan 08	Mar 08	Aug 08
Mean	-0.32	9.64	-1.14	-2.58
Std. dev.	6.58	4.60	4.28	4.43
# of days > 0	52	119	52	34
# of days < 0	77	7	71	93

Table 8.4.3.: EUETS case study: Properties of estimators. Means, standard deviations and number of positive and negative days of the four estimators. Measured in Euros.

	Nov 07	Jan 08	Mar 08	Aug 08
Ljung-Box	867.68	738.84	1606.87	797.01
χ^2 (95%)	36.06	35.73	35.40	35.84

Table 8.4.4.: EUETS case study: Testing the estimator for white noise. Ljung-Box statistics and 95% quantiles of the chi-squared distribution.

are now going to conduct Ljung-Box white noise tests for each series. Results are shown in Table 8.4.4. The null hypothesis (i.e. series is white noise) is rejected for all levels and all four estimators. We conclude that the estimators are significantly not white noise and proceed to test the measurability.

8.4.1.2. The Second Phase of the EUETS: Regression Results

We will now use the regression-based approach to check the non-measurability of $\hat{I}_{\mathcal{G}}^{\mathbb{Q}}(t, T_1, T_2)$. Looking at the graphs of the auto-correlation functions we suspect stationary first differences as announced in Section 8.2. We formally justify using first differences by applying the Dickey-Fuller test of stationarity. Results are shown in Table 8.4.5:

The Dickey-Fuller test rejects stationarity at all relevant levels for the pure time series while it accepts stationarity at all levels for first differences.

As discussed in Section 8.2.2, the basis that we now use is the polynomial basis of the first spot differences, i.e. $\{\triangle S^i, i \in \mathbb{N}\}$. Therefore, we consider the regression

$$\triangle \hat{I}_{\mathcal{G}}^{\mathbb{Q}}(t, T_1, T_2) = \sum_{i=1}^{N} c_i \triangle S_t^i + \triangle \epsilon_t$$

The results of this regression with $N = 10$ are given in Table 8.4.6 and some supplementary material is located in Section C.1. Here, the critical value of the

X	S	Nov 07	Jan 08	Mar 08	Aug 08
DF(X)	-0.84	-0.68	-0.13	-0.25	-0.31
DF($\triangle X$)	-19.98	-7.23	-7.68	-8.37	-6.80

Table 8.4.5.: EUETS case study: Testing for stationarity.. Dickey-Fuller statistics of the spot and estimators and their corresponding first differences.

	Nov 07	Jan 08	Mar 08	Aug 08
R^2	0.14	0.07	0.03	0.07
F-statistic	1.47	0.65	0.35	0.75

Table 8.4.6.: EUETS case study: Regression results.. R^2 and F-statistics of the regression from the first differences of the spot to those of the estimators.

F-distribution at the 95% level for all four data sets is 1.88 and thus for all four series the hypothesis of zero value coefficients is accepted (still, results for November 2007 are less obvious, as expected). For the January 2008 contract all individual coefficients have zero value and t-statistics are insignificant at all levels so that none of the basis polynomials possesses any explanatory power. We find similar results for the March 2008 forward. Table C.1 gives the corresponding figures. The November 2007 forward, i.e. the forward chosen to deliver before the keydate of 01/01/2008 exhibits less straightforward numbers. Some coefficients are significant and p-values are small (cf. Table C.2). This is the expected behaviour and we would conclude measurability to some degree. We remark that, as mentioned before, results do not change with a larger N, i.e. when adding more basis monomials.

Summarising, we can conclude that for contracts traded on 01/01/2008 the spot does not help to explain the variations of the residual term. We have thus proved that there exists a substantial part of the forward price which is orthogonal to the historical filtration and which consequently is non-measurable. Results for other contracts are mixed and partially significant (and we refer to Section 8.5 for further discussions). We therefore claim that the non-measurable estimators are indeed information premia induced by the beginning of the second phase of the EUETS.

8.4.1.3. The Second Phase of the EUETS: Discussion

Now that our test has confirmed that the estimator $\hat{I}^{\mathbb{Q}}_{\mathcal{G}}(t, T_1, T_2)$ satisfies the properties of the information premium for contracts traded during the critical time period we will discuss results and also provide some more qualitative insights. Figure 8.4.8 again shows the information premium for the crucial forward with delivery period in January 2008. It is positive over nearly the whole lifetime of the contract with a mean of 9.64 Euros (i.e. 14% of the average forward price of 65 Euros). The second important feature is that the premium is relatively stable over the first 70 days (with a variance of 2.31), followed by a more turbulent period (variance 8.43 from day 71 until the beginning of January). Considering the extra information about the introduction of the emission certificates we would have expected this type of behaviour. Clearly the market hypothesised an upward shift in electricity prices due to higher costs for emission certificates. This upward shift corresponds to the positive graph of the residual. During January, i.e. the delivery period, this additional information is then step by step incorporated into the historical filtration and the information premium thus tends to zero.

Let us discuss the value of the January 2008 information premium shortly. The EEX spot price for CO_2-certificates during the second half of 2007 was practically

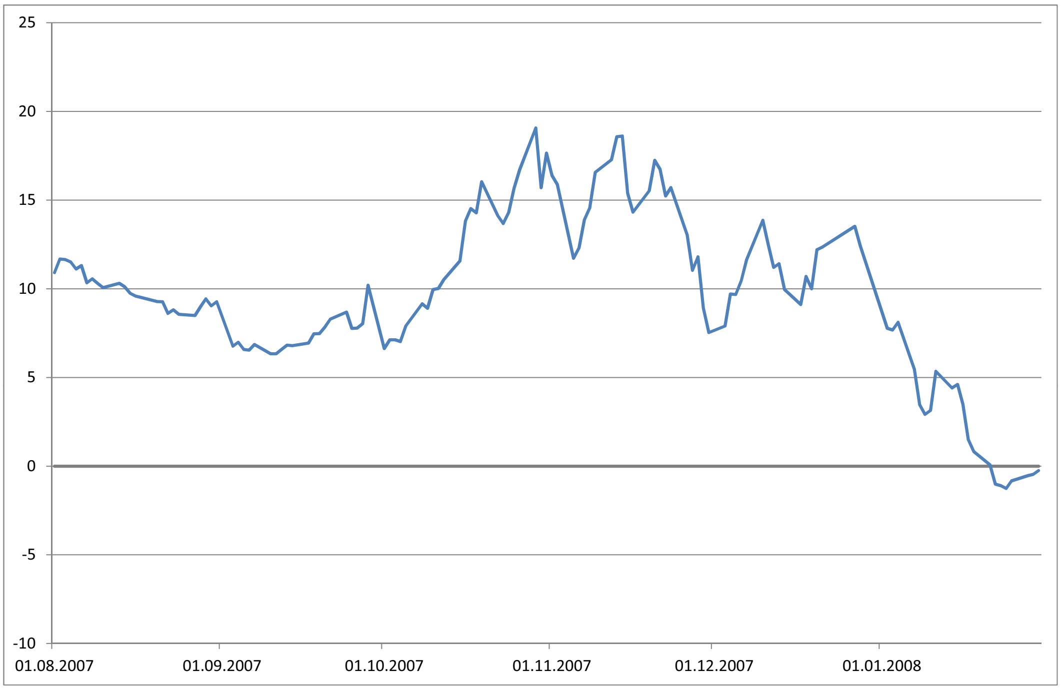


Figure 8.4.8.: EUETS case study: Information premium January 2008. As identified and confirmed by the testing procedure. In Euros.

zero (less than 8 cents, cf. market data at EEX (2012a) or Ströbele et al. (2010, Figure 14.1)). This was due to a number of reasons. Firstly, the first (trial) phase of the EUETS suffered from severe over-allocation that resulted in the collapse of prices at the beginning of 2007 (when certified emission figures were first released, cf. Burger et al. (2007, Section 1.5.4) or Ströbele et al. (2010, Section 14.4.1) for discussions). Secondly, market design did not include a banking property for permits so that horded certificates would expire worthless in 2008 (cf. Burger et al. (2007, page 40)). Still, at the same time the forward price for the year 2008 (i.e. the beginning of phase two) was between 18.50 Euros in August 2007 and 23.80 Euros in November 2007, with an average of around 22.00 Euros for one tonne of CO_2. For the year 2007 a report of the Umweltbundesamt (2012) gives the average German CO_2-intensity factors for production and consumption of electricity at 0.608 and 0.629 tCO_2/MWh. The small difference is to be explained by imports and exports.

Multiplying these efficiency factors with the price bounds of CO_2-forwards mentioned above we would expect extra costs for electricity between 11.25 Euros and 14.97 Euros with a mean of 13.38 Euros. Compared to the information premium in Figure 8.4.8 these are, in fact, very similar figures for the upshift. Hence, also the size of our information premium is in line with our economic intuition.

Before we continue with the second case study, we would like to refer the reader to the working paper and presentation of Trück (2012). There, the author discusses emission markets and their impact on electricity prices in Australia and introduces a so-called *carbon premium* which he uses in addition to the risk premium. We remark

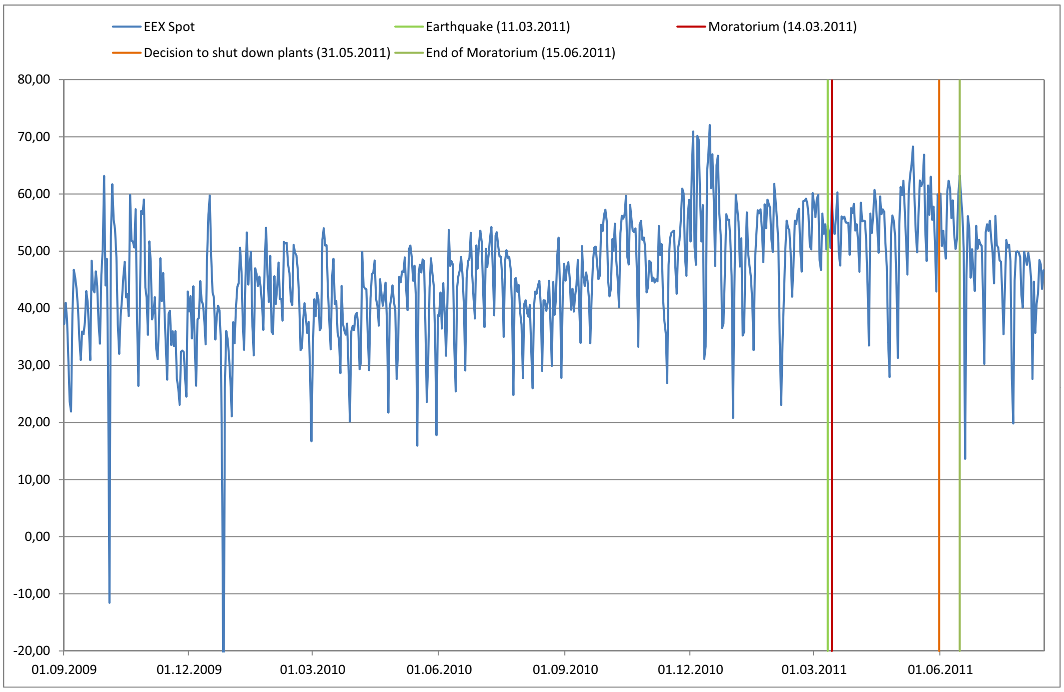


Figure 8.4.9.: Moratorium case study: Spot price. EEX data from 01/09/2009 until 15/08/2011. Vertical bars denote key-dates. In Euros.

that for our current case studies and the additional future information considered in this section, the carbon premium is the information premium in disguise.

8.4.2. The German Atom Moratorium

Now, we will discuss the second case study as motivated in Section 1.2.2: the German *Atom Moratorium*. Again, electricity data used is taken from the German EEX and the time range considered in this section is 01/09/2009 until 15/08/2011, which are 711 days. The critical dates are 11/03/2011 (earthquake), 14/03/2011 (Moratorium), 31/05/2011 (final decision to close old plants) and 14/06/2011 (the end of the Moratorium). Figure 8.4.9 shows the spot price whereas calibration results as well as a comparison between observed and simulated moments are provided in Section B.3. We have already mentioned in the introduction that the spot did not react to the Moratorium but forward prices did and hence, we suspect the existence of an information premium.

Analysing the spot price more closely perhaps the most striking feature of this newer dataset, remembering the spot ranges in Section 3.7, Section 8.3 or Section 8.4.1, is that there are hardly any spikes. Those spikes still existent mostly have negative heights rather than positive ones (the calibration identifies almost 90% of negative spikes, cf. Table B.6 for concrete parameter values). It is obvious that there have been several structural changes on the German electricity market. Overall liquidity and market design have improved, prices are less volatile and less

Forward	1 m	2 m	3 m	4 m	5 m	6 m
θ_W	0.210	0.624	0.650	0.614	0.512	0.363

Table 8.4.7.: Moratorium case study: Girsanov parameters. For different forward classes.

spiky. Again, we can check Table B.6 and find a σ of only around 6.0 and a λ of 2.7% (compared to 8.4 and 7% in the EUETS dataset). The reason for negative price spikes is the massive increase in the production of renewable energy. Electricity thus produced has to be used by law and enjoys a guaranteed tariff. Overall, in 2011, 20.3% of German electricity was generated by renewables, an increase of 3% compared to 2010 (cf. the report of the Bundesumweltministerium (2012, page 12)). Strong wind or extreme sun will therefore lead to a huge price decline, especially in times of low demand. An example is Christmas 2009 which featured a daily average price of less than -20 Euros. We refer to Wagner (2012) for a comprehensive study and detailed modelling of renewable energy and its impact on the German market.

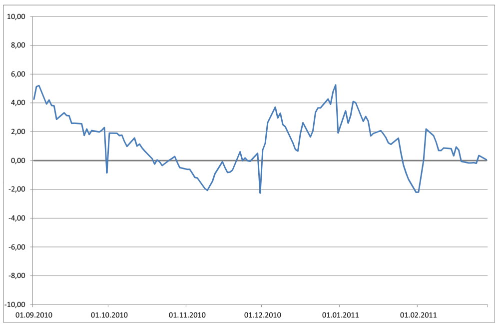
We conduct a measure change in the same way as before and the corresponding parameters are given in Table 8.4.7. Again, fundamental changes in the risk-adjustment of market traders are clearly visible. We no longer experience the change of sign between measure change parameters with short and long times until delivery. Instead, parameters take small positive values for all classes of forwards. Remembering the arguments of Benth et al. (2008a) and Section 8.4.1 it now seems that with a decreasing spike intensity, less volatility and higher efficiency of the market the hedging pressure of both retailers as well as producers has decreased, too.

8.4.2.1. The German Atom Moratorium: Estimator

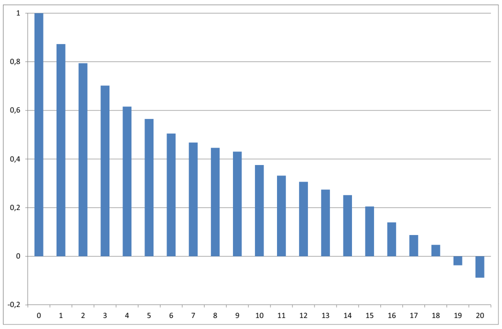
We can now calculate expectations and forward prices under the real-world and the pricing measures. As before, we will again examine specific forward contracts. We have chosen one benchmark contract unaffected by the Moratorium with delivery in February 2011 and the two forwards with delivery in May 2011 and July 2011.

The resulting estimators $\hat{I}^{\mathbb{Q}}_{\mathcal{G}}(t, T_1, T_2)$ and corresponding auto-correlation functions are illustrated in Figure 8.4.10.

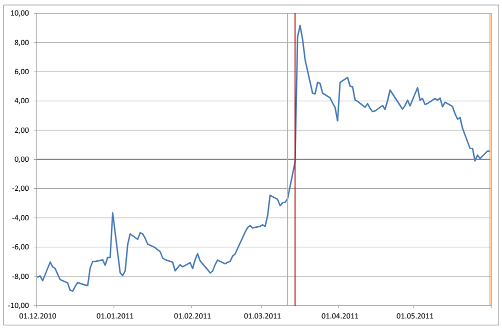
The price impact of the Moratorium for the May and July contracts can clearly be seen in Subfigure 8.4.10c as well as Subfigure 8.4.10e. Both graphs exhibit a huge upward shift in prices on 14/03/2011 which we claim constitutes an information premium. The estimator of the February (cf. Subfigure 8.4.10a) is much less eye-catching. It changes sign frequently and takes smaller values in general and here we would doubt the existence of a significant information premium. Important properties of the three residuals are collected in Table 8.4.8. The estimator for May 2011 has a much higher standard deviation and is also clearly divided into two parts: before the Moratorium (mean of -6.36 Euros) and after (mean 3.82 Euros). We have a similar situation for July 2011 with a mean of 6.54 Euros during the three months of the Moratorium and one of -0.64 Euros before and after. Still, it is surprising that the July estimator returns to its pre-Moratorium level even before



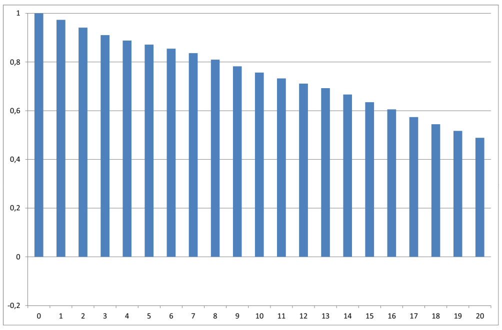
(a) February 2011: Information premium est.



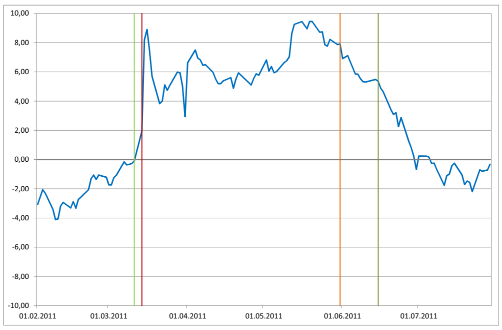
(b) February 2011: acf of estimator



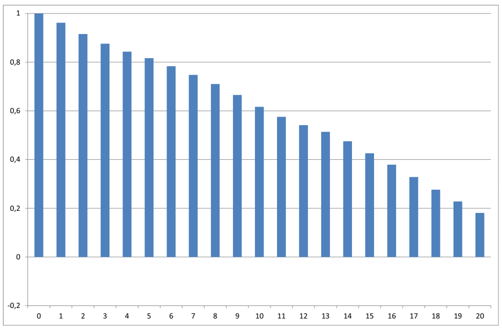
(c) May 2011: Information premium est.



(d) May 2011: acf of estimator



(e) July 2011: Information premium est.



(f) July 2011: acf of estimator

Figure 8.4.10.: Moratorium case study: Estimators. And auto-correlation functions. Vertical bars denote key-dates. In Euros.

	Feb 11	May 11	Jul 11
Mean	1.27	-2.07	2.92
Std. dev.	1.76	5.36	4.05
# of days > 0	93	53	77
# of days < 0	36	75	48

Table 8.4.8.: Moratorium case study: Properties of estimators. Means, standard deviations and number of postitive and negative days of the three estimators. Measured in Euros.

	Feb 11	May 11	Jul 11
Ljung-Box	522.97	1645.87	1094.83
χ^2 (95%)	36.06	35.95	35.62

Table 8.4.9.: Moratorium case study: Testing the estimator for white-noise. Ljung-Box statistics and 95% quantiles of the chi-squared distribution.

the beginning of the delivery period. We will discuss this issue in Section 8.4.2.3. The auto-correlation functions once again suggest non-stationary but also indicate stationary first differences.

According to our test method we need to confirm the non-zero property. Once again, we apply the Ljung-Box test for white noise. Results are shown in Table 8.4.9. The null hypothesis is rejected for all relevant levels and all three forwards.

8.4.2.2. The German Atom Moratorium: Regression Results

We will now use our regression test method to check the measurability properties of the estimators $\hat{I}^{\mathbb{Q}}_{\mathcal{G}}(t, T_1, T_2)$ isolated in Section 8.4.2.1. Once again, the Dickey-Fuller test of stationarity is applied to both the time series under consideration as well as their first differences. Results are summarised in Table 8.4.10. Here, the critical value for the 10%-level is approximately -1.61. Thus, the Dickey-Fuller test rejects stationarity at all relevant levels for the pure time series while it accepts stationarity at all relevant levels for first differences.

As before and as motivated in Section 8.2.2, we will use the polynomial basis of the spot price $\{\triangle S^i, i \in \mathbb{N}\}$. The regression formula is then:

X	S	Feb 11	May 11	Jul 11
DF(X)	-0.48	-0.95	-0.28	-0.21
DF($\triangle X$)	-21.46	-9.97	-6.86	-6.52

Table 8.4.10.: Moratorium case study: Testing for stationarity. Dickey-Fuller statistics of the spot and estimators and their corresponding first differences.

	Feb 11	May 11	Jul 11
R^2	0.14	0.06	0.09
F-statistic	1.94	0.69	1.09

Table 8.4.11.: Moratorium case study: Regression results. R^2 and F-statistics of the regression from the first differences of the spot to those of the estimators.

$$\triangle \hat{I}_{\mathcal{G}}^{\mathbb{Q}}(t, T_1, T_2) = \sum_{i=1}^{N} c_i \triangle S_t^i + \triangle \epsilon_t$$

The results of this regression with ten basis elements are displayed in Table 8.4.11. Here, the critical values of the F-distribution at the 90% and 95% levels are 1.67 and 1.91 respectively. Thus, the hypothesis of zero coefficients is accepted for the May and July forwards, although not for the February contract. Values and t-statistics of the individual coefficients for May 2011 and February 2011 are listed in Section C.2. Table C.3 and Table C.4 generally confirm insignificant coefficients for both contracts but yield stronger and more unambiguous p- and t-values for May 2011. Hence, our test confirms the existence of a non-measurable information premium for the May 2011 and July 2011 contracts but rejects its existence with mixed results for the February 2011 contract.

8.4.2.3. The German Atom Moratorium: Discussion

We are now ready to analyse what happened after the Moratorium was enacted. There are two questions that remain to be answered. We need to find out why the spot did not at all react to the shut-down of 8 GW of cheap nuclear energy. Also, we still need to explain the peculiar shape of the information premium of the July 2011 forward in comparison to that of May 2011.

Considering the first question, there were a number of reasons for the way the market developed. Firstly, due to the season there was a lot of spare capacity. Secondly, two of the plants, Brunsbüttel and Krümmel (both in Schleswig-Holstein, capacity of 800 and 1400 MW), had previously suffered from constant maintenance problems and were already offline since 2007 and 2009, respectively. A third reactor (Biblis B in Hessen, 1300 MW) had gone offline earlier in 2011 for regular maintenance. This reduced the capacity to be shut down during the week following the new legislation by more than three GW. Thirdly, and rather accidentally, more (cheap) wind energy was produced in Germany (cf. Bundesnetzagentur (2011, page 13)).

Another more important reason why spot prices did not increase was the change in import and export. While Germany exported 4000 MW before the Moratorium, it started importing 1000 MW after, mainly of course, from French nuclear power plants.

Again, details on renewables and cross-boarder trading in the context of the Moratorium can be found in Bundesnetzagentur (2011, page 15 ff.). Summarising, there

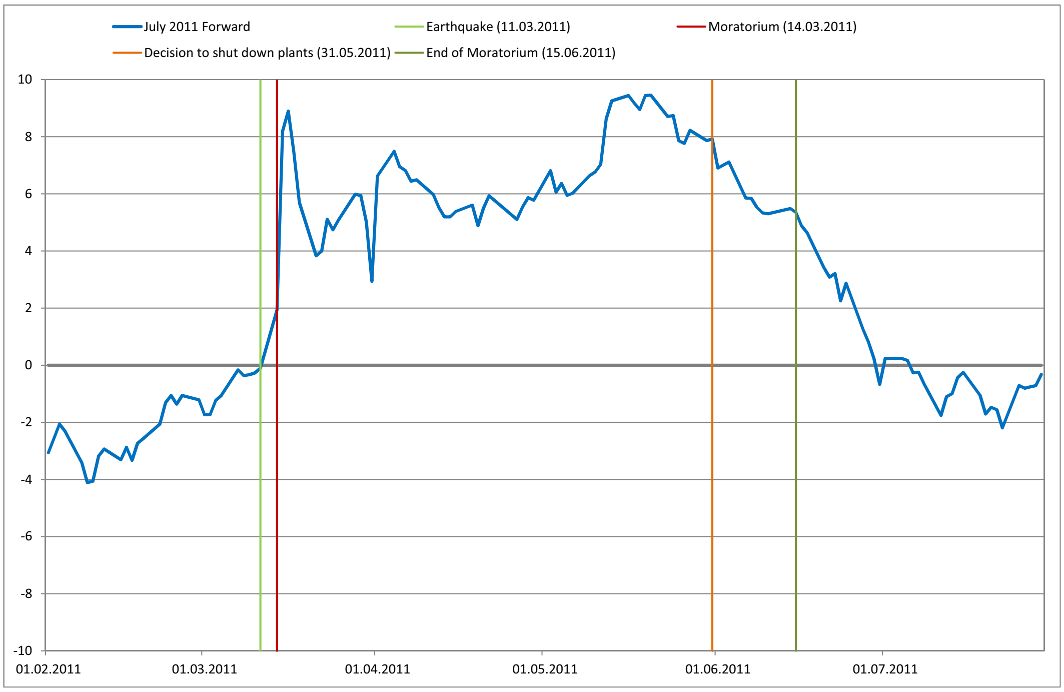


Figure 8.4.11.: Moratorium case study: Information premium July 2011. As identified and confirmed by the testing procedure. In Euros.

was no shift towards a more expensive fuel in the merit order and thus no immediate increase in the spot price.

Concerning forward prices we have already explained the information premium for May 2011 (we refer to Section 1.2.2 and Section 8.4.2.1). After the Moratorium we exhibited a persisting price up-shift. The market obviously expected a shift in the merit order as cheap nuclear energy had to be replaced. Thus, we will now concentrate on the premium for July 2011, which is illustrated in Figure 8.4.11. After being close to zero for one and a half months it jumps upwards following the Moratorium oscillating around 6.00 Euros (i.e. 11% of the average forward price of 53.50 Euros). Still, even after the final decision to shut down the seven old plants (and before the beginning of the delivery period), the residual tends back to zero. We remark that this illustrates nicely that the information premium is a function in time. It seems that market sentiment about the consequences of the new policies changed over the lifetime of the forward. What was the reason for this behaviour? Firstly, in addition to the plants mentioned above, another four reactors went into regular maintenance with the beginning of May (cf. Bundesnetzagentur (2011, page 42)). This reduced the available (cheap) nuclear capacity even further and finally caused the change of the marginal plant in the merit order. Secondly, also political uncertainty was removed from the market with the decision for a permanent shut-down. Generally, this led to a more relaxed market situation after May and a vanishing information premium.

Forward	MA(2)	MA(4)	MA(7)	MA(10)	MA(30)
January 08	0.10	0.09	0.08	0.09	0.08
November 07	0.17	0.16	0.16	0.16	0.16

Table 8.5.12.: Further regressors: Moving average. R^2 of the regression with polynomial basis and moving averages of different length.

8.5. Further Regressors and Robustness of Test

There are a number of issues concerning our method and the case studies that extend our discussion or still need consideration. These will be covered in this section. The interplay between additional information, our estimator for the information premium and further time series will be explored in Section 8.5.1. Furthermore, we will check robustness properties of our test by means of simulations in Section 8.5.2.

8.5.1. Further Regressors

8.5.1.1. Moving Averages of the Spot

As the historical filtration does not only represent today's information but also that of the past, one might claim that we have only showed the residual to be non-Markovian in the spot price. Thus, we will now add moving averages of the spot of different lengths to the list of regressors. This will include some idea of past spot knowledge. The regression function is then

$$\triangle \hat{I}_{\mathcal{G}}^{\mathbb{Q}}(t, T_1, T_2) = c_1 MA_l(S_t) + \sum_{i=2}^{N+1} c_i \triangle S_t^i + \epsilon_t$$

where l denotes the length of the moving average; we tried lengths of $l = 2, 4, 7, 10, 30$ days. Table 8.5.12 shows the R^2 for the forwards of the EUETS dataset. Compared to our findings in Section 8.4.1.2 we do not see too large a difference. Indeed, for both data sets (the EUETS and the Moratorium sets) we found the general result was not altered and the coefficients of the moving average were insignificant.

8.5.1.2. Gas as a Marginal Fuel

Another question that needs to be answered is whether our test does reject non-measurability when applied to a storable commodity. An example could be one of the (marginal) fuels such as coal or gas. Thus, we will now apply our test in the following manner: we will regress from polynomials of the EEX gas spot price to the estimator of the May 2011 forward of the Moratorium dataset. The price of the EEX gas spot is illustrated in figure Figure 8.5.12.

It is easy to see that the gas price obviously reacted to the announcement of the Moratorium by an increase of 1.50 Euros (which is the second biggest price increase

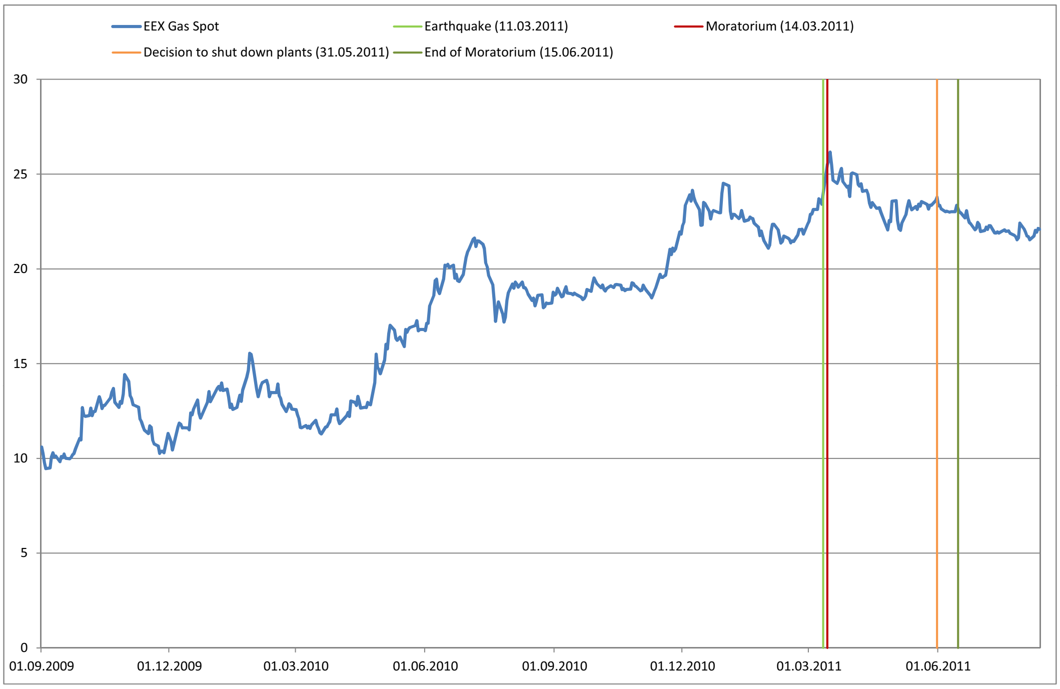


Figure 8.5.12.: Further regressors: EEX gas spot. For the Moratorium dataset, i.e. from 01/09/2009 until 15/08/2011. In Euros.

	Feb 11	May 11	Jul 11
R^2	0.26	0.18	0.18
F-statistic	4.13	2.57	2.43

Table 8.5.13.: Further regressors: Gas regression results. R^2 and F-statistics of the regression from the first differences of the gas spot to those of the estimators.

for one day in the time series). We will regress the following equation:

$$\triangle \hat{I}_{\mathcal{G}}^{\mathbb{Q}}(t, T_1, T_2) = \sum_{i=1}^{N} c_i \triangle GAS^i(t) + \triangle\epsilon(t)$$

The results for $N = 10$ and the forwards from the last section are summarised in Table 8.5.13 and Table C.5. We do not only see larger values for R^2 and significant F-statistics but also huge values for individual coefficients as well as a mixture of significant and insignificant t-statistics. Hence, we would reject the existence of a large information premium and conclude that the additional information (here, the Moratorium) was already priced into the gas spot. This clearly makes sense as gas is storable for example in storage facilities and to some degree even in the pipeline network.

	Feb 11	May 11	Jul 11
R^2	0.13	0.63	0.74
F-statistic	1.80	19.79	8.21

Table 8.5.14.: Further regressors: DAX regression results. R^2 and F-statistics of the regression from the first differences of the DAX index to those of the estimators.

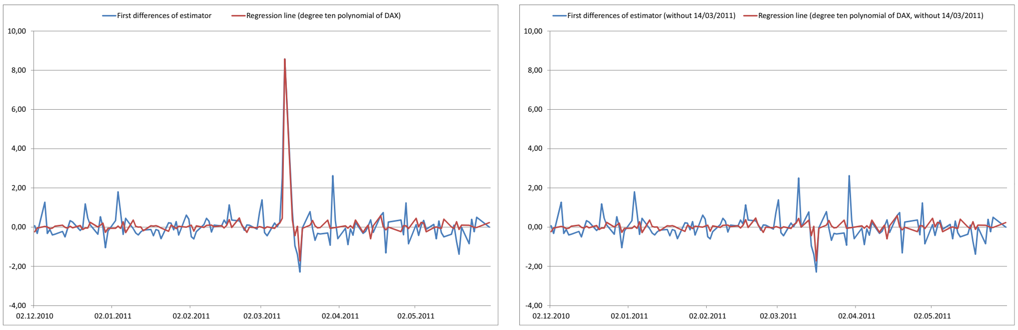
8.5.1.3. The German DAX

Last but not least, we will now use the German stock index DAX as a regressor. Thereby we will check that the information premium identified in the last section was actually caused by the Moratorium and not some other influence of the economy. Also, we will point out a peculiar market situation. We conduct the following regression:

$$\triangle \hat{I}_{\mathcal{G}}^{\mathbb{Q}}(t, T_1, T_2) = \sum_{i=1}^{N_1} c_i \triangle DAX_t^i + \triangle \epsilon_t$$

Table 8.5.14 provides the surprising R^2 and F-statistics for the three contracts. While the February forward confirms the usual very low correlation between DAX and electricity prices we face a totally different picture for May and July 2011. This is also reflected in the t-statistics of individual coefficients, cf. Table C.6. Here, we would have to reject the non-measurability property. The reason for this peculiar result can be found in the week after the earthquake. The earthquake itself occurred on a Friday whereas the Moratorium and its consequent rise in forward prices took place on Monday and Tuesday thereafter. Prices decreased to some extend during the week as market participants realised more clearly the consequences for the German electricity market. For example, the price of the May 2011 forward was 50.88 Euros on Friday, it jumped to 61.95 Euros on Tuesday and settled to a level around 58.00 Euros by the end of that week. Exactly the opposite took place on the stock exchange. On Friday, the DAX was at 6981 points. The stock exchange closed for the weekend and when it reopened on Monday and Tuesday, traders had a first impression of the long-term damages and their impact on the Japanese economy. Consequently, the DAX fell by more than 400 (i.e. 5.7%) to 6513 points. Still, less than two weeks later the DAX had regained all losses. These extreme losses/gains of only a few days are responsible for the high R^2 and the significance of the result.

Summarising, two (almost independent) events caused the jumps in electricity forward prices and the DAX index. This is further illustrated in Figure 8.5.13 which shows the resulting regression function for polynomials of DAX residuals up to degree ten compared to the differences of the estimator. In Subfigure 8.5.13b the regression line is shown with values of 14/03/2011 replaced by zero. Changing only the data of this day reduces the R^2 from 63% to 19% which would allow to conclude non-measurability, as expected.



(a) Regression function. With 14/03/2011. **(b) Regression function**. Without 14/03/2011.

Figure 8.5.13.: Further regressors: DAX regression results. First differences of the estimator regression function with and without 14/03/2011.

8.5.2. Robustness

To check robustness of our test (i.e. to check whether it behaves as expected) we will now conclude this chapter by conducting a simulation study. To this end, we will assume that the spot price evolves according to a standard Gaussian Ornstein-Uhlenbeck process with constant parameters

$$dS_t = \alpha(\mu - S_t)dt + \sigma dW_t^1$$

where W_t^1 is a standard Brownian motion. We define another process Z^1 by

$$dZ_t^1 = -\alpha_{Z^1} Z_t^1 dt + \sigma_{Z^1} dW_t^{Z^1}$$

where $W_t^{Z^1}$ is another Brownian motion which is independent of W_t^1. Let W_t^2 be yet another Brownian motion and let $dW_t^1 dW_t^2 = \rho dt$ be the correlation coefficient. Then we define the process Z_t^2 by

$$dZ_t^2 = -\alpha_{Z^2} Z_t^2 dt + \sigma_{Z^2} dW_t^2$$

Now, we construct the forward price as follows:

$$F(t) = \mu + p_1 Z_t^1 + p_2 Z_t^2$$

where $p_1, p_2 \in [0, 1]$ are constants.

The motivation behind this construction is as follows: we have seen before that due to the large rate of mean reversion calculated forward prices tend to be almost constant in terms of t. This is why we choose constant μ to be the first building block of the forward price. We note that with a constant change of measure this value is not very important. Both processes Z_t^1 and Z_t^2 are Ornstein-Uhlenbeck processes around zero and resemble the random shocks in the forward price. Z_t^1 is independent of S_t and may be interpreted as part of the information premium or some other non-measurable deviations. Z_t^2 on the other hand, is an Ornstein-Uhlenbeck process induced by the spot price according to ρ. Depending on the parameters but in

ρ	1.0	0.8	0.5	0
$p_1 = 0, p_2 = 0.5$	0.98	0.64	0.28	0.05
$p_1 = p_2 = 0.5$	0.51	0.34	0.17	0.05

Table 8.5.15.: Robustness of test: Simulation results 1/2. R^2 for different values of p_1, p_2 and ρ for 1000 simulations.

p_2	0	0.1	0.25	0.5	1.0
R^2	0.05	0.09	0.24	0.52	0,80

Table 8.5.16.: Robustness of test: Simulation results 2/2. R^2 for different values of p_2. $p_1 = 0.5$ and $\rho = 1.0$. 1000 simulations.

particular on the choice of ρ, p_1, p_2 we expect to see similar regression results as in Section 8.4 but also clear rejections of the existence of an information premium (similar to what we found out in Section 8.5.1.2). In the following, we will use the polynomial basis of degree ten and compare mean statistics of 1000 simulations.

We choose parameters similar to those extracted from market data or which make sense economically, respectively. We set $\alpha = 0.5$ and $\sigma = 5.0$. Furthermore, we set $\alpha_{Z^2} = 0.3$ and $\sigma_{Z^2} = 3.0$ as we expect forward prices to be less volatile. We set $\alpha_{Z^1} = 0.3$ and $\sigma_{Z^1} = 3.0$, too. Mean values of R^2 for some combinations of ρ, p_1 and p_2 are illustrated in Table 8.5.15.

As expected, for $p_1 = 0$, regressing the forward differences on the polynomials of spot differences yields high R^2s for high coefficients of correlation with a near-zero R^2 for zero correlation. Concerning significance, only the value of the coefficient of $\triangle S_t$ is significantly different from zero (as indeed expected by construction). All F-statistics except those for a zero ρ reject zero coefficients. For $p_1 = 0.5$ we get smaller values for R^2 as this implies the introduction of orthogonal random shocks to this simplified framework.

In order to classify the results of the previous sections, we are now going to consider a series of experiments in which we will assume $\rho = 1.0$ and modify the value of p_2 ceteris paribus. The question we would like to answer by conducting this experiment is what type of setup would yield results of the same quality as in Section 8.4.1 or Section 8.4.2. Table 8.5.16 provides the facts and figures. For example, remembering the forward with delivery in January 2008 our regression had an R^2 of 0.07 (cf. Table 8.4.6). Roughly speaking, looking at Table 8.5.16, this would correspond to the situation in which one sixth of variations (i.e. $p_1 = 0.5$ and $p_2 = 0.1$) of the residual were induced by the spot whereas the rest was induced by some non-measurable other source. This seems sensible for our hypothesised information premium around that time.

Summarising, by this simple framework of a simulation experiment we have shown two things: Firstly, we can replicate results from previous sections and their case studies. Secondly, we have extended the findings of Section 8.5.1.2 in so far as we have constructed non-trivial scenarios for which the hypothesis of non-measurability is rejected. This would correspond to the case for which our test would reject the existence of an information premium.

8.6. Contribution and Discussion

This chapter is the empirical centrepiece of the dissertation at hand. It features the first empirical investigation dealing with the impact of information asymmetry on electricity markets in general and the information premium in particular. There are various contributions to the academic literature.

We propose a statistical test that provides an estimator for the information premium and checks the desired properties. The most important of these inherent properties is its non-measurability with respect to the space spanned by the historical filtration. Dealing with this orthogonality is consequently the most difficult aspect of the method presented. Our test is based on Hilbert space representation and regression and can be generally applied as no assumptions on the precise structure of the market filtration are made. When it comes to financial applications, this means that not only the underlying electricity might be of interest here but potentially every good that is difficult to store or not completely storable. Mathematically, to the best of our knowledge, there is no other literature on how to test for the measurability of different objects empirically. Generally speaking, given two time series our test is able to tell whether or not one is measurable with respect to the space spanned by the other. This might point towards a whole variety of future applications.

We explore the robustness of the test in two different ways. Firstly, we apply a simulation approach that confirms that our test returns the right answer depending on the current scenario construction. Secondly, the results we find when additionally taking into consideration a storable commodity (gas) and a rather uncorrelated stock index (DAX) are consistent.

Furthermore, by providing estimators for the information premium during the two different market scenarios (the beginning of the second phase of the EUETS and the German Atom Moratorium) this chapter also illustrates the relevance and importance of the new spot-forward relationship, in particular when it comes to pricing forwards and futures. The shape of our estimators confirms the economic intuition and helps to get a quantitative insight into certain market mechanisms.

To prepare for the test we conduct a parametric distance-minimising measure change, thereby being able to verify or falsify previous results from the literature on the risk premium. For the newer Moratorium data we find that the typical structure of the risk premium seems to have changed ever since most of the relevant literature was published. This quantitatively confirms recent and rather qualitative findings, in particular studies on the impact of renewable energy and the corresponding lack of positive spikes in a maturing electricity market.

Concluding, this chapter does not only provide the missing empirical investigation and thus explain the great relevance of this dissertation but it should also strongly encourage the consideration of our new spot-forward relationship for electricity markets.

Appendices

A. Miscellaneous

We use the following theorem in the proof of Theorem 2.2.1. Its proof can be found in Protter (2005, Theorem 4).

Theorem A.1. Stricker's theorem. *Let X be a semi-martingale under the filtration $\mathcal{G}$ and let $\mathcal{F}$ be a subfiltration of $\mathcal{G}$ such that X is adapted with respect to $\mathcal{F}$. Then X is an $\mathcal{F}$-martingale.*

Theorem A.1 of Benth and Meyer-Brandis (2009) is a slightly simplified version of Théorème I.1.1 of Chaleyat-Maurel and Jeulin (1985). It is provided for completeness next.

Theorem A.2. Chaleyat-Maurel-Jeulin theorem. *Let $g(t)$ be a twice-integrable function and define for a Brownian motion W_t the filtration*

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma\left(\int_0^\infty g(s)dW_s\right)$$

Furthermore, define auxiliary function

$$\gamma(t) = \int_t^\infty g^2(s)ds$$

Then

$$\xi_t = W_t - \int_0^t \int_s^\infty \frac{g(s)g(u)}{\gamma(s)} dW_u ds$$

is a Brownian motion under filtration $\mathcal{G}_t$.

B. Calibration Results for Chapter 8

B.1. The Dataset from 10/2003 until 10/2006

Table B.1, Table B.2 and Table B.3 show some calibration results for the EEX spot price from 06/10/2003 until 26/10/2006 as discussed in Section 8.3.2.

Param.	α	σ	β	λ	p	q	η_1	η_2
Est.	0.317	7.96	0.519	0.015	0.929	0.071	0.020	0.054

Table B.1.: EEX spot price 10/2003 until 10/2006: Estimated parameters. Of stochastic components.

	Mean	Std. Dev.	Skewness	Kurtosis
Orig.	39.38	17.75	1.94	6.21
Sim.	40.30	18.31	1.53	12.17

Table B.2.: EEX spot price 10/2003 until 10/2006: Estimation quality. Based on 5000 simulations.

Forward	1 m	2 m	3 m	4 m	5 m	6 m	...	6 q
θ_W	-0.205	-0.260	-0.576	-0.986	-1.377	-1.735	...	-6.354

Table B.3.: EEX spot price 10/2003 until 10/2006: Girsanov parameters. For different classes of month- and quarter-forward contracts.

B.2. The EUETS Data Set

Table B.4 and Table B.5 show some calibration results for the EEX spot price during the EUETS case study as discussed in Section 8.4.1.

B.3. The Moratorium Data Set

Table B.6 and Table B.7 show some calibration results for the EEX spot price during the Moratorium case study as discussed in Section 8.4.2. We remark that the high variance of the parameter p is due to the reason that some simulations do not have a single positive jump. Therefore, this parameter takes a zero value for these paths resulting in large deviations.

C. Regression Results for Chapter 8

C.1. The EUETS Data Set

As before, Table C.1 and Table C.2 provide additional results for the regressions of Section 8.4.1.2.

C.2. The Moratorium Data Set

Again, Table C.3 and Table C.4 provide additional results for the regressions of Section 8.4.2.2.

Param.	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
Est.	21.19	0.093	4.213	-1.09	-2.88	-0.52	5.574	2.950	-7.09	-16.97
Var.	1.370	0.004	0.935	0.905	0.977	0.975	0.760	0.775	0.767	0.760
Param.	b_{10}	b_{11}	b_{12}	α	σ	λ	p	η_1	η_2	β
Est.	4.495	5.909	5.129	0.566	8.437	0.070	0.756	0.029	0.029	0.763
Var.	0.771	0.761	0.761	0.077	0.443	0.008	0.096	0.003	0.005	0.050

Table B.4.: EUETS case study: Spot price parameters. EEX spot price and variance of 5000 simulations. From 01/02/2007 until 30/10/2008.

	Mean	Std. Dev.	Skewness	Kurtosis
Orig.	52.18	23.94	0.69	0.27
Sim.	52.65	25.17	0.51	5.06

Table B.5.: EUETS case study: Original and simulated moments. Based on 5000 simulations.

Param.	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
Est.	37.93	0.024	0.942	1.490	-1.059	1.330	3.497	3.276	2.660	1.427
Var.	0.940	0.002	0.666	0.626	0.663	0.680	0.477	0.466	0.477	0.472
Param.	b_{10}	b_{11}	b_{12}	α	σ	λ	p	η_1	η_2	β
Est.	-4.28	-8.466	1.881	0.499	6.008	0.027	0.105	0.046	0.033	0.864
Var.	0.475	0.474	0.472	0.066	0.248	0.006	0.111	0.012	0.006	0.066

Table B.6.: Moratorium case study: Spot price parameters. EEX spot price and variance of 5000 simulations. From 01/09/2009 until 15/08/2011.

	Mean	Std. Dev.	Skewness	Kurtosis
Orig.	45.71	10.39	-1.18	6.12
Sim.	45.26	11.51	-2.27	19.65

Table B.7.: Moratorium case study: Original and simulated moments. Based on 5000 simulations.

	Value	Std. Dev.	t-value	p-value
$\triangle S^1$	-0.0079	0.0318	-0.2493	0.8037
$\triangle S^2$	-0.0009	0.0021	-0.4066	0.6853
$\triangle S^3$	0.0000	0.0002	0.2090	0.8349
$\triangle S^4$	0.0000	0.0000	0.2463	0.8060
$\triangle S^5$	0.0000	0.0000	-0.1727	0.8633
$\triangle S^6$	0.0000	0.0000	-0.0220	0.9825
$\triangle S^7$	0.0000	0.0000	0.1130	0.9103
$\triangle S^8$	0.0000	0.0000	-0.1329	0.8946
$\triangle S^9$	0.0000	0.0000	-0.0678	0.9461
$\triangle S^{10}$	0.0000	0.0000	0.0970	0.9229

Table C.1.: EUETS case study: January 2008 regression. Results for individual coefficients of the regression: $\triangle \hat{I}^{\mathbb{Q}}_{\mathcal{G}}(t, T_1, T_2) = \sum_{i=1}^{10} c_i \triangle S^i_t + \epsilon_t$.

	Value	Std. Dev.	t-value	p-value
$\triangle S^1$	-0.1223	0.0679	-1.8022	0.0748
$\triangle S^2$	-0.0061	0.0057	-1.0757	0.2849
$\triangle S^3$	0.0012	0.0007	1.7188	0.0890
$\triangle S^4$	0.0000	0.0000	1.9302	0.0567
$\triangle S^5$	0.0000	0.0000	-1.6746	0.0975
$\triangle S^6$	0.0000	0.0000	-2.4302	0.0171
$\triangle S^7$	0.0000	0.0000	1.6443	0.1036
$\triangle S^8$	0.0000	0.0000	2.4088	0.0180
$\triangle S^9$	0.0000	0.0000	-1.6008	0.1129
$\triangle S^{10}$	0.0000	0.0000	-1.9613	0.0529

Table C.2.: EUETS case study: November 2007 regression. Results for individual coefficients of the regression: $\triangle \hat{I}_{\mathcal{G}}^{\mathbb{Q}}(t, T_1, T_2) = \sum_{i=1}^{10} c_i \triangle S_t^i + \epsilon_t$.

	Value	Std. Dev.	t-value	p-value
$\triangle S^1$	0.1223	0.1261	0.9699	0.3341
$\triangle S^2$	-0.0035	0.0275	-0.1271	0.8991
$\triangle S^3$	-0.0089	0.0153	-0.5826	0.5613
$\triangle S^4$	0.0005	0.0018	0.2854	0.7758
$\triangle S^5$	0.0001	0.0006	0.1851	0.8535
$\triangle S^6$	0.0000	0.0000	-0.3145	0.7537
$\triangle S^7$	0.0000	0.0000	0.0113	0.9910
$\triangle S^8$	0.0000	0.0000	0.2933	0.7698
$\triangle S^9$	0.0000	0.0000	-0.1037	0.9176
$\triangle S^{10}$	0.0000	0.0000	-0.2507	0.8025

Table C.3.: Moratorium case study: May 2011 regression. Results for individual coefficients of the regression: $\triangle \hat{I}_{\mathcal{G}}^{\mathbb{Q}}(t, T_1, T_2) = \sum_{i=1}^{10} c_i \triangle S_t^i + \epsilon_t$.

	Value	Std. Dev.	t-value	p-value
$\triangle S^1$	-0.0959	0.0795	-1.2059	0.2303
$\triangle S^2$	-0.0144	0.0197	-0.7319	0.4657
$\triangle S^3$	0.0052	0.0079	0.6660	0.5067
$\triangle S^4$	0.0005	0.0011	0.4388	0.6616
$\triangle S^5$	-0.0001	0.0002	-0.5900	0.5563
$\triangle S^6$	0.0000	0.0000	-0.2614	0.7942
$\triangle S^7$	0.0000	0.0000	0.5085	0.6120
$\triangle S^8$	0.0000	0.0000	0.2182	0.8276
$\triangle S^9$	0.0000	0.0000	-0.4274	0.6699
$\triangle S^{10}$	0.0000	0.0000	-0.2341	0.8153

Table C.4.: Moratorium case study: February 2011 regression. Results for individual coefficients of the regression: $\triangle \hat{I}_{\mathcal{G}}^{\mathbb{Q}}(t, T_1, T_2) = \sum_{i=1}^{10} c_i \triangle S_t^i + \epsilon_t$.

	Value	Std. Dev.	t-value	p-value
$\triangle Gas^1$	2.2902	0.8931	2.5644	0.0116
$\triangle Gas^2$	2.9856	2.5191	1.1852	0.2384
$\triangle Gas^3$	-12.8089	8.8565	-1.4463	0.1508
$\triangle Gas^4$	-14.3295	16.8076	-0.8526	0.3957
$\triangle Gas^5$	28.2032	23.5003	1.2001	0.2326
$\triangle Gas^6$	16.6196	33.7947	0.4918	0.6238
$\triangle Gas^7$	-23.0508	22.0313	-1.0463	0.2976
$\triangle Gas^8$	-6.2221	26.1148	-0.2383	0.8121
$\triangle Gas^9$	6.0177	6.4503	0.9329	0.3528
$\triangle Gas^{10}$	0.4616	6.6865	0.0690	0.9451

Table C.5.: Further regressors: Gas and May 2011 regression results. Results for individual coefficients of the regression: $\triangle\hat{I}^{\mathbb{Q}}_{\mathcal{G}}(t,T_1,T_2) = \sum_{i=1}^{10} c_i \triangle Gas^i_t + \epsilon_t$.

	Value	Std. Dev.	t-value	p-value
$\triangle DAX^1$	0.0053	0.0042	1.2379	0.2183
$\triangle DAX^2$	0.0000	0.0001	0.3494	0.7274
$\triangle DAX^3$	0.0000	0.0000	-1.7018	0.0915
$\triangle DAX^4$	0.0000	0.0000	-0.7353	0.4637
$\triangle DAX^5$	0.0000	0.0000	1.7163	0.0888
$\triangle DAX^6$	0.0000	0.0000	1.1605	0.2482
$\triangle DAX^7$	0.0000	0.0000	-1.7265	0.0869
$\triangle DAX^8$	0.0000	0.0000	-1.5073	0.1345
$\triangle DAX^9$	0.0000	0.0000	1.7754	0.0785
$\triangle DAX^{10}$	0.0000	0.0000	1.7395	0.0846

Table C.6.: Further regressors: DAX and May 2011 regression results. Results for individual coefficients of the regression: $\triangle\hat{I}^{\mathbb{Q}}_{\mathcal{G}}(t,T_1,T_2) = \sum_{i=1}^{10} c_i \triangle DAX^i_t + \epsilon_t$.

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